The microwave course – Exercises

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A. Impedance, admittance, and LC-resonators

For a sufficiently low frequency (and we are yet to build up our understanding of how low is that), the dynamics of electromagnetic field can be reduced to a network of inductors and capacitors. We denote both elements as a box with two wires sticking out of it, with the understanding that an L-labeled box means inductance and a C-labeled box means capacitance. A voltage and a current source are denoted by a big circle with a wavy line in the middle. And ideal voltage source V has zero series resistance and an ideal current I source has an infinite parallel resistance. We also agree to consider all signal to have a time-dependence of the form $V(t) = \hat{V} \exp i\omega t$ and $I(t) = \hat{I} \exp i\omega t$. The amplitudes $\hat{V}$ and $\hat{I}$ are themselves complex numbers, and the real voltages and currents are obtained by taking the real part of their complex-valued functions. This way we reduce differentiation in the time domain by a multiplication by $i\omega$, which simplifies the math dramatically. The quantity connecting V to I is called impedance and is indicated as $Z(\omega)$. The inverse of impednace is what connected I to V, it is called admittance, and is usually denoted as $Y(\omega)$. For example, impedance of an inductor is $Z_L = i\omega L$ and admittance of a capacitor is $Y_C = i\omega C$.

With these notations in mind, we can formally define an inductance as an element, which results in a current $\hat{I}$ when biased by an ideal voltage source $\hat{V}$ such that $\hat{V} = (i\omega L)\hat{I}$. Likewise, for a capacitor we would get $\hat{V} = (1/i\omega C)\hat{I}$. In both cases, the complex numbers $\hat{V}$ and $\hat{I}$ are represented by vectors perpendicular to each other. In case of inductance, $\hat{V}$ is "ahead" of $\hat{I}$ by 90 degrees, and it's the opposite for the case of capacitance (we remind that $\pm i = \exp(\pm i\pi/2)$). For the sake of simplicity of notations we will often skip the "hat" in the complex number notation for voltages and currents.

Connecting an inductor to a capacitor results in the formation of an LC circuit which has a resonance at the frequency $\omega_0 = 1/\sqrt{LC}$. There are two ways to arrange such a connections. First (see Fig. 1a,b is connecting a voltage source to a series combination of L and C). In this case, the total impedance connected to the voltage source is $Z(\omega) = i\omega L - i/\omega C$ and it becomes zero at the frequency $\omega_0 = 1/\sqrt{LC}$. That is, for a finite AC-voltage amplitude applied, the resulting current has a diverging amplitude. That's a resonance of current, or a series resonance. The second way of connecting an inductor to a capacitor is shown in Fig. 1c,d. This time a current source is connected to a parallel combination of L and C, which is equivalent to an admittance $Y(\omega) = i\omega C - i/\omega L$. At frequency ω_0 we have the admittance going to zero, that is a finite amplitude current source induces a diverging voltage. That's a resonance of voltage or the parallel LC-resonance. So, a series resonance is characterized by the zero of the impedance function of the entire circuit and a parallel resonance is characterized by the zero of the admittance function.

The divergences are eliminated in the presence of some dissipation. In case of the parallel LC-circuit, a small dissipation rate can be introduced by adding a high-value resistor R in parallel with L and C (see Fig. 1e). Now admittance is never zero for a real frequency. However, it does have a zero but for complex valued frequency. $\hat{\omega} \approx \omega_0 + i\epsilon$, where for a large enough R we should get $\epsilon \ll \omega_0$.

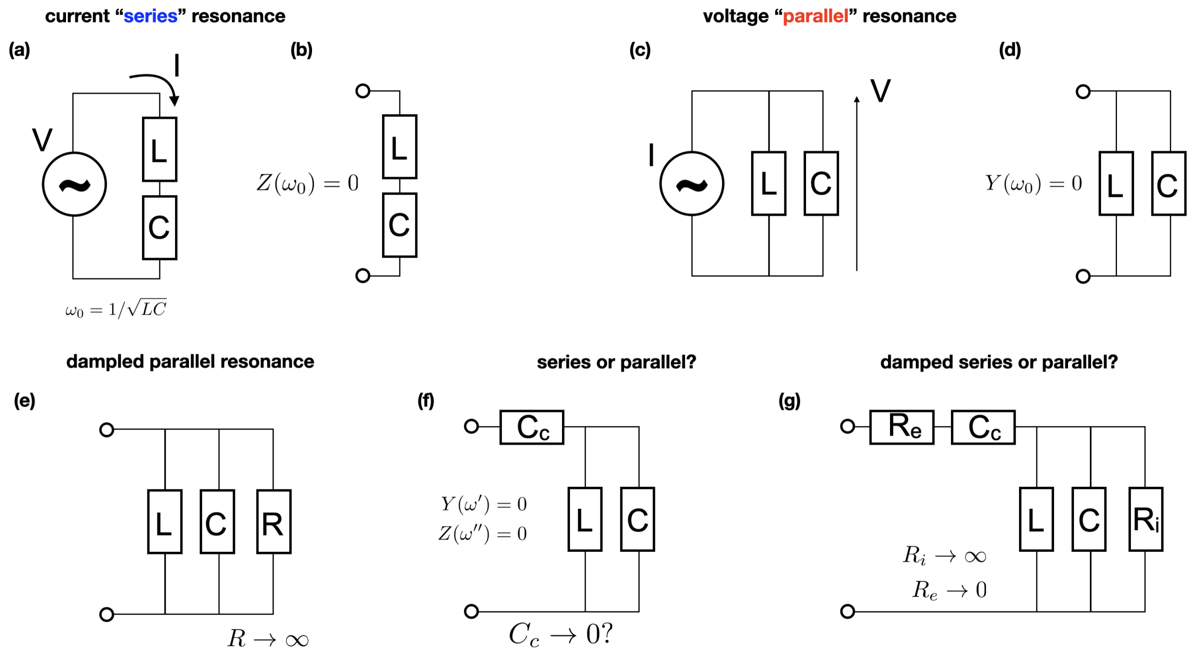


Figure 1: Understanding resonances via impedance and admittance functions

Exercise 1.1 Find a condition on R for which $\epsilon \ll \omega_0$, calculate ϵ and connect its value to the quality factor Q of the resonance. What do you think constraints the sign of ϵ?

Let us minimally complicate the case of LC resonances by adding a third element, this time a small capacitance $C_c \ll C$ connected as shown in Fig. 1f. Is this a series or a parallel resonance circuit?

Exercise 1.2 A little paradox. Consider a circuit in Fig. 1f. Find ω' such that $Z(\omega') = 0$ and ω'' such that $Y(\omega'') = 0$. Do not hesitate to make Taylor series expansion in C_c to the lowest order to simplify the answer. Plot both functions vs frequency. So, are we looking at a series or a parallel resonance? What happens to ω' and ω'' when we take the limit of a very small capacitance C_c?

Exercise 1.3 Let's add some loss (see Fig. 1g). Resistor R_i represents the so-called "internal" dissipation, say due to losses in the capacitor C or inductance L or wires connecting them, and resistor R_e models an "external" dissipation, say due to the internal resistance of the voltage source. With the choice of the circuit in Fig. 1g, the internal loss becomes negligible when R_i is very large (open circuit) and the external loss becomes negligible when R_e is very small (short circuit). So let's fix some value of $C_C << C$, calculate $Z(\omega)$ and $Y(\omega)$, and explore their zeros. Describe the resonance behavior. Here you can use both the analytical approach or Microwave Office. Is there still a paradox when we send C_C towards zero?

B. Transmission lines

Transmission lines is an electromagnetic structure with a cylindrical geometry, conceptually consisting of two independent conductors "parallel" to each other, in the sense that the cross-section of the line is the same along the line. Such a line propagates electromagnetic excitations qualitatively similar to plane waves in vacuum, also known as TEM (transverse electric magnetic). The energy of the TEM wave resides mainly in the vicinity of the line conductors in the form of spatially confined electric and magnetic fields. A telegraph is the first example of such a TEM transmission line. A USB cable or a coaxial cable is a more modern implementation of the "telegraph". There is also a plenty of on-chip or printed circuit board (PCB) transmission lines with geometries known as microstrip (the first conductor is a strip, the second is the "ground" plane), coplanar waveguide (CPW) (a 2D projection of a coaxial cable), coupled strips (two microstrip lines parallel to each other), etc.

Interestingly, the vast zoo of TEM transmission lines can be modeled by a relatively simple ladder network of inductors and capacitors (Fig. 2). The model is motivated by the physical structure of the line: there is electric field between the lines due to the potential difference, and there is magnetic field winding around the lines due to the currents. Instead of electric and magnetic fields we focus on currents and voltages, while Maxwell's equations and boundary conditions are replaced by the Kirchoff's laws. We define a voltage wave as a sum of forward and backward propagating waves

$$V(x) = V_+ \exp(ikx) + V_- \exp(-ikx), \tag{1}$$

where both the propagation constant k and the amplitudes V_+ and V_- generally depend on the frequency ω. Likewise, we define a current wave

$$I(x) = I_+ \exp(ikx) + I_- \exp(-ikx) \tag{2}$$

For the simplest ladder network shown in Fig. 2c, we would get

$$\begin{aligned} &k = \omega/v \\ &v = 1/\sqrt{lc} \\ &V_+/I_+ = V_-/I_- = Z_\infty = \sqrt{l/c} \end{aligned} \tag{3}$$

where we used the inductance l and capacitance c per unit length of the line $L = l \times dx$ and $C = c \times dx$. The quantity v is the phase velocity and the quantity Z_∞ is the wave impedance. Interestingly, the quantities c and l depend weakly on the line geometry and, in case there is now media near the conductors, a good estimate is $c \approx \epsilon_0$ and $l \approx \mu_0$. The deviation of $k(\omega) = \omega/v$ law is called dispersion, it means some frequencies propagate faster than the others, and hence it can significantly alter the propagation of signals involving m and it can significantly distort the propagation of signals containing multiple frequencies. A purely imaginary k would mean propagation is forbidden (why?).

Exercise 1.4 Let's consider a more general ladder model for a transmission line, where the "inductance" part is given by an element with a frequency-dependent impedance $Z_1(\omega)$ (per unit length) and the "capacitance" part is given by an element with a frequency-dependent admittance $Y_2(\omega)$. Such more general model is physically justified, because as the wire gets longer, impedance adds up but admittance adds up as well and this can take into account the fact that ideal inductance and capacitance model might be an oversimplification. The deviation of $k(\omega) = \omega/v$ law is called dispersion and it can significantly alter propagation of waves consisting of multiple frequencies.

Show that for such a general transmission line $k(\omega) = \sqrt{-Z_1(\omega)Y_2(\omega)}$ and $Z_\infty(\omega) = \sqrt{Z_1(\omega)/Y_2(\omega)}$

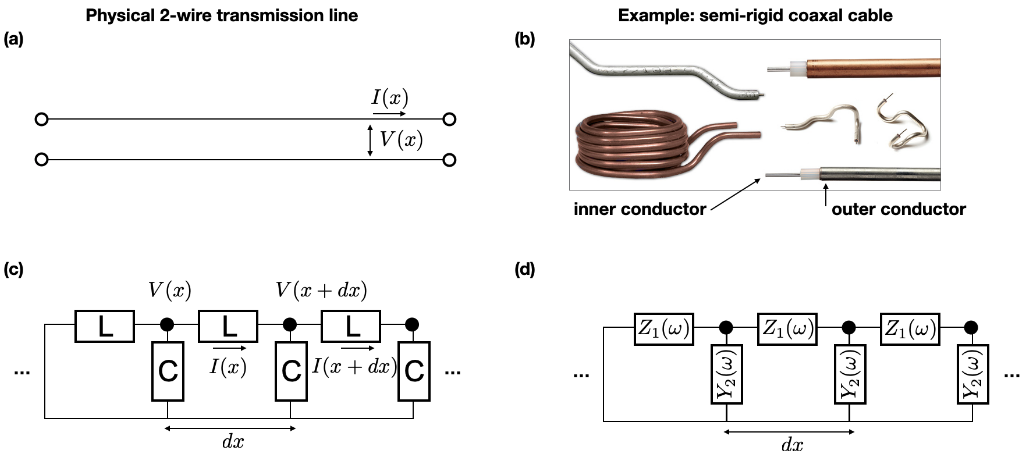
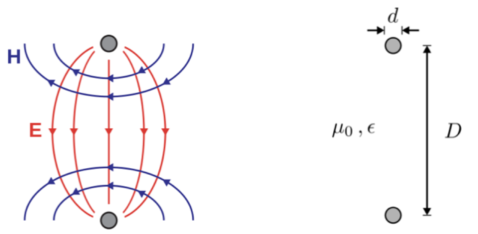


Figure 2: TEM transmission lines

Exercise 1.5 Consider a case where $Z_1(\omega)$ represents a parallel resonance circuit consisting of capacitance C' and inductance L. Consider then a different case where $Y_1(\omega)$ represents a parallel resonance circuit consisting of capacitance C and inductance L'. Analyze the properties of wave propagation in such transmission lines in low- and high-frequency limits.

Exercise 1.6 Consider a case where $Z_1(\omega) = i\omega L + r(\omega)$ and calculate the propagation constant $k(\omega)$. A frequency-dependent resistor can be used to model the skin-effect. Namely for a conductor with a conductivity σ, the AC current flows only in the surface "skin"-layer $\delta_s = 1/\sqrt{\mu_0 \sigma \omega}$. That is, as the frequency goes up, the skin layer gets thinner, so the same amplitude current would dissipate more energy (less conductor cross-section is available). Therefore, while a good model for the inductor of a wire is $L = \mu_0 \times dx$, a good model for the resistance is $r(\omega) = dx \times (1/\sigma) \times 1/(\pi d \delta_s)$, where d is the wire diameter. Thus, for conductors much wider than δ_s, we get $r(\omega) \propto \sqrt{\omega}$. At low enough frequencies the current flows in the entire wire cross-section, so r is just the wire's resistance. At high enough frequencies, the skin-depth becomes so small that the Drude model of conductivity requires modifications. For a good conductor, δ_s is on the order of a few microns at 1 GHz, so the skin effect plays an important role in the microwave frequency range for conductors thicker than a few microns. Estimate the loss per unit length in a coaxial cable made of Copper inner conductor with a diameter of 0.5 mm. Plot your result vs frequency in the 1 - 20 GHz range.



$$L = \frac{\mu_0}{\pi} \ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]$$

$$C = \frac{\pi\epsilon}{\ln\left[(D/d) + \sqrt{(D/d)^2 - 1}\right]}$$

Figure 3: Cross-section of a two-wire transmission line and the effective ladder inductance and capacitance per unit length

Exercise 1.7 Refer to Fig. 3 for an accurate model for the ladder inductance and capacitance in a two-wire transmission line. Assume $\epsilon = \epsilon_0$. Calculate the speed of light. Does it depend on the geometry of the line? What about wave impedance? Design a geometry such that $Z_\infty = 1\ \Omega, 100\ \Omega, 1000\ \Omega, 10000\ \Omega$. Any problems with such a transmission line design request?

C. Terminated transmission lines and impedance transformations