Control and operation of tokamaks
Exercise 0 - Prerequisites

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1 State Space representation

a)Write down the set of ODEs defining the circuit dynamics.

$$0 = L\frac{di}{dt} + Ri + e$$
$$i = C\frac{de}{dt}$$

b) Write them as a first order ODE in a vector matrix form.

$$\frac{de}{dt} = \frac{i}{C}$$
$$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{L}e$$

c)Among the variables, identify the states of the system.

One possible choice is e, voltage across the capacitor and i, current flowing the circuit.

d) Write down the state space representation for this system.

$$\dot{X} = AX + BU,$$
$$Y = CX + DU$$

$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}, B = \begin{bmatrix} 0 \end{bmatrix} C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} D = \begin{bmatrix} 0 \end{bmatrix}$$

With no forcing $B = 0$ and with no feedthrough D=0.

e)Write down the state space representation of this new system

$$\dot{X} = AX + BU,$$
$$Y = CX + DU$$

$$A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This time $B \neq 0$ while $D = 0$. The input is a voltage and the outputs are the states of the system.

f) Determine the stability of the system

```
%% State space representation
A = [0,1/Cc;-1/Lc,-Rc/Lc]; % Construct A matrix
B = [0;1/Lc]; % Construct B matrix
C = [1,0;0,1]; % Construct C matrix (The states are the outputs)
D = [0;0]; % Construct D matrix
sys = ss(A,B,C,D); % ss construct the state space representation

%% Stability of the system
figure(1);clf;
pzmap(sys) % Displays the poles and zeros of the system
pole = max(real(pole(sys))); % Gets the pole on the real axis
fprintf('The pole location on the real axis %d',pole)
```

The poles of the system always feature $Re(p) < 0$ therefore the system is stable. This is physically justified by the presence of only energy conserving or dissipating elements in the system.

g) Plot the step response for the various inductance values.

```
step(sys(:,1)); % Step response at the voltage
input to the outputs (states of the system)
```

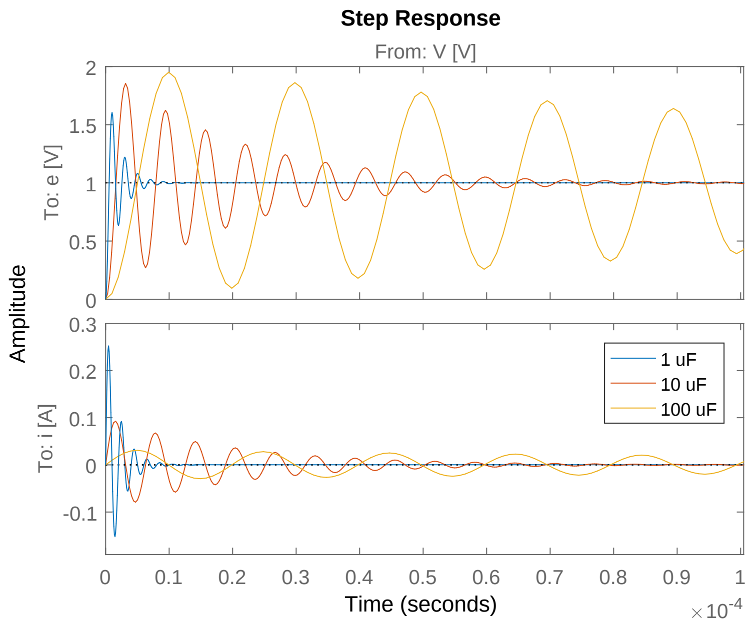

Figure 1: Step responses for various inductances.

Exercise 2 - Circuit diagram (Electrical Engineering)

a) Derive the transfer function from voltage to current for this circuit.

$$V = L\frac{di}{dt} + Ri$$

Laplace transform yields the following equation

$$V(s) = Lsi(s) + Ri(s)$$

The transfer function is as follows:

$$G(s) = \frac{i(s)}{V(s)} = \frac{A}{\tau s + 1}$$

where, $A = 1/R$ is the steady state gain, $\tau = L/R$ is the time constant.

b) Plot the step response for a $1V$ step for $R = 1\Omega$ and $L = 1\mu H$, $L = 10\mu H$, and $L = 100\mu H$

```
%% Transfer function representation
Ag = 1/Rc; % Steady state gain
tau = Lc/Rc; % Time constant
s = tf('s'); % Define s as a transfer function variable
```

```
G = Ag/(tau*s+1);% Transfer function for the system v(s) to i(s)

%% Step response using the transfer function model
figure(3);clf;
step(G); % Step response at the voltage input to the current
```

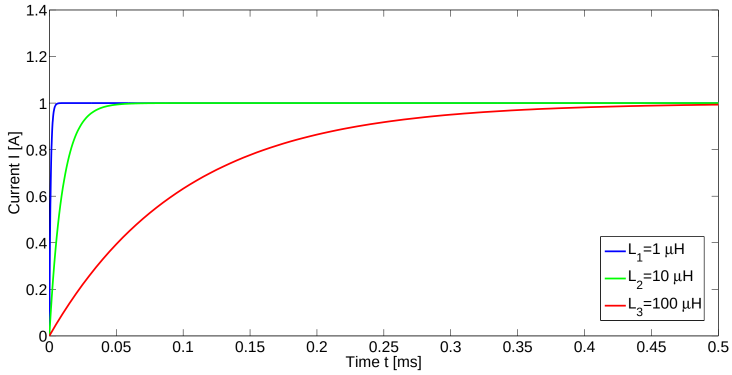

Figure 2: Step responses for various inductances.

c) Sketch the bode plot for the different inductances. Compare them by using the matlab bode function. Use the Bode diagrams to explain the step responses.

```
%% Bode plot for the transfer function
figure(4);clf;
bode(G); % Bode plot for the transfer function
```

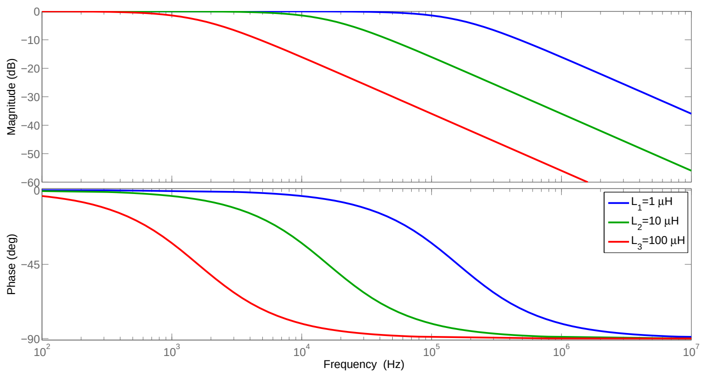

Figure 3: Bode plots for various inductances.

d) Convert the transfer function to a discrete time plant with Tustin mapping. Note that it is a rational function of z^-1. Study its stability with various settings

```
Ts = 1e-7; % Sample time for the discrete system,
this has to less than the system time constant
Gd = c2d(G,Ts,'tustin');
figure(5);clf;
step(Gd)
figure(6);clf;
impulse(Gd); % Impulse response
```

Exercise 3 - P-controller design (Control)

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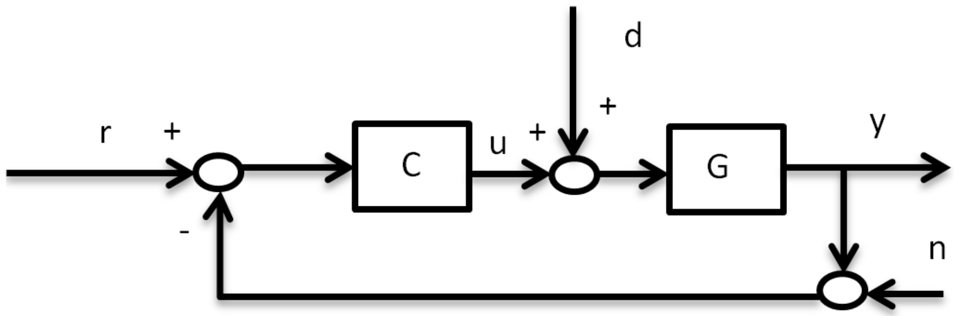

Figure 4: Control scheme for exercise 3.

a) For the plant $G(s)$, design a proportional controller that provides a maximum bandwidth (frequency up to which $\mid G(s)C(s) \mid \geq 1$), while ensuring stability.

This question is subtle. The plant G is a stable second order system (it features two poles and no zero) and stability with a proportional controller is in principle ensured for ANY (positive) gain of the controller C. This can be verified by observing that the phase ϕ of the system transfer function never moves below $-\pi$ and this allows to always satisfy the Bode stability criterion for a stable plant in feedback (positive phase margin $PM = \phi - (-\pi)$ of the open loop system $L = CG$ at the bandwidth). Note that this does not imply that ROBUST stability is achieved, since the stability margins get extremely low for a high K.

b) Verify both conditions (Maximum sensitivity function and Nyquist criterion) for your controller. Taking $C(s) = K$, the maximum bandwidth is achieved at maximum K that still satisfies the stability margin. Using Matlab, one can find iteratively $K = 1.4$. The plots for the proportional controller are obtained using the Matlab commands:

```
K=1.4;
C=K;
figure
subplot(221)
nyquist(P*C)
axis(3*[-1 1 -1 1])
hold all
% margin
theta=0:1/100:2*pi;
plot(-1+0.5*cos(theta),0.5*sin(theta),'r')
title('Nyquist diagram')

subplot(222)
bodemag(1/(1+P*C),w)
hold on
plot([w(1) w(end)],[6 6],'r--')
title('Sensitivity function')

subplot(223)
step(1-1/(1+P*C))
title('Step on reference')
subplot(224)
step(P/(1+P*C))
title('Step on disturbance')
```

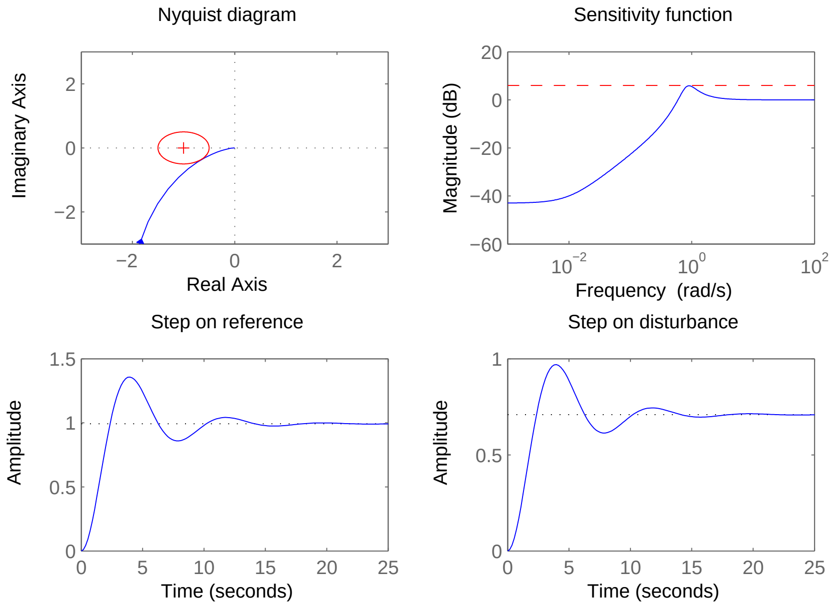

Figure 5: Results for proportional controller.

The Nyquist diagram in figure 5 shows that the system passes the circle with radius 0.5 on the right, indicating stability. The sensitivity is nicely below the 6dB. Note in the step responses that one oscillation takes approximately 6 seconds, this corresponds to the bandwidth of approximately 1 rad/s. Note that the proportional controller is not able to remove a disturbance at the plant input.

c) Derive the transfer function between the u(t) and r(t), y(t) and r(t), d(t) and y(t).

$$\frac{y}{r} = \frac{GC}{1+GC} \qquad \text{closed loop}$$

$$\frac{u}{r} = \frac{C}{1+GC} \qquad \text{control sensitivity}$$

$$\frac{y}{d} = \frac{G}{1+GC} \qquad \text{process sensitivity}$$

d) Plot the step response for the above derived transfer functions.

Refer the figure 5.

e) Add a delay to the controller response u(t) and plot the root locus for the closed loop system and study the effect on the stability of the system.

```
%% Adding a delay to the controller
C_delay = ss(C); % State space for the controller
C_delay.InputDelay = 0.6; % Delay for the controller
C_del = pade(C_delay); % Pade approximation
L_del = C_del*G;
L = C*G;
figure(1); rlocus(L) % root locus of the open loop
figure(2); rlocus(L_del) % root locus of the open loop with delay
```

The root locus shows how the poles change following a variation of the controller gain in feedback. The poles of the closed loop system start from the open loop ones and move away from them. For the system with no delay they never get unstable (i.e. they never move in the half-plane with positive real part). In the case with delay instead they cross the imaginary axis for a sufficiently high gain. Since it's impossible to have a signal transmission without delay, it will be impossible to reach an infinite bandwidth.

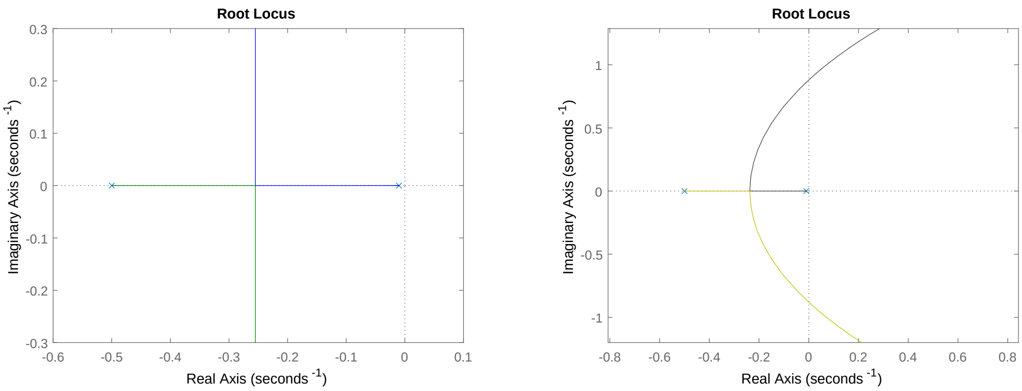

Figure 6: a) Root locus with no delay. b) Root locus in presence of delay