Magnetic modeling and control of tokamaks, Part III: Plasma position stability and control

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EPFL

Outline I

Section 1

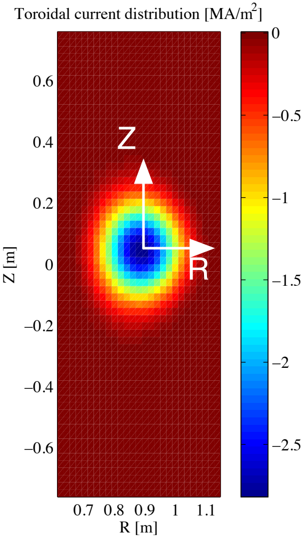
Control of plasma position

Subsection 1

Force balance

Rigid plasma description

- So far we assumed the plasma was magically held in place, however the plasma does not stay in place by itself.
- We will now extend our simple model and assume the plasma is a rigid conductor that can (rigidly) move around in the poloidal plane. Its center has a position (r, z) which may vary in time.
- We give a qualitative picture here and refer to MHD courses/books for details e.g. [1, 2]



Radial force balance - tire tube force

- The p term is (also called the **tire tube force**) comes from the fact that the surface of the plasma on the low field side is smaller than the surface at the high field side. Assuming the pressure is constant on the surface, the net force is $\mathbf{F_{tire}} = \mathbf{e_R}(pS_{out} - pS_{in})$ and since $S_{out} > S_{in}$, $\mathbf{F_{tire}}$ is directed outward.

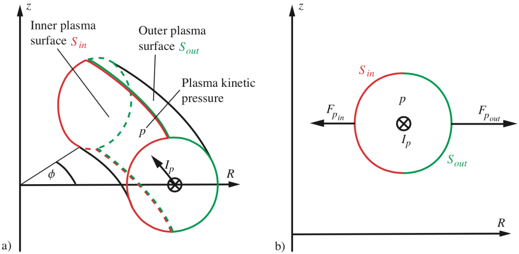


Figure: Tire force on a tokamak plasma [3]

Radial force balance - hoop force

- The B_θ term (also called the **hoop force**) comes from the fact that the poloidal flux is compressed on the high-field side, so $|B_{p,1}| > |B_{p,2}|$. Any toroidal current feels a Lorentz force that will be outward on the high field side and inward at the low field side. $B_p \sim I_p$, the lorentz force $\sim B_p^2$. Taking the surface into account we get $F_h \sim \mathbf{e_R}(B_1^2 S_1 - B_2^2 S_2)/(2\mu_0)$.

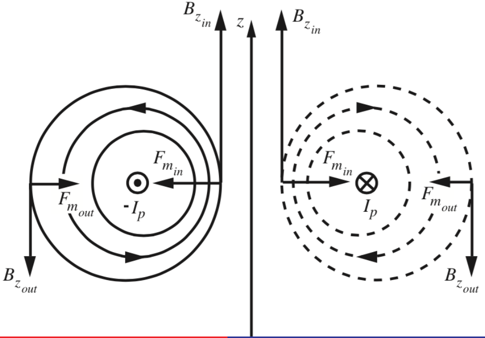


Figure: Hoop force on a tokamak plasma [3]

Radial force balance - $1/R$ force

- The $1/R$ **force** comes from the fact that the toroidal field $B_\phi \sim \frac{1}{R}$ is stronger on the high-field side, so $|B_{\phi,1}| > |B_{\phi,2}|$.
- For a diamagnetic plasmas (poloidal plasma currents reduce B_ϕ), there will be an outward force.
- For paramagnetic plasmas (poloidal plasma currents increase B_ϕ) there will be an inward force.

The need for a vertical field

- It can be shown [4, 2] that an approximate expression for the the total outward force is:

$$F_R = \frac{\mu_0 I_p^2}{2} \underbrace{\left(\ln \frac{8 R_p}{a\sqrt{\kappa}} + \beta_p + \frac{\ell_i}{2} - \frac{3}{2} \right)}_{\Gamma} \quad (1)$$

  where R_p, a are the plasma major and minor radius, κ is the elongation, β_p is a measure for the plasma pressure and ℓ_i is the normalized internal inductance $\ell_i = 2L_p/(R_p\mu_0)$ (to be treated later in the course). This equation is an approximation valid in the limit $R_p/a >> 1$.

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- To counteract this radial force, we impose a **vertical field $\mathbf{B}_v$** inside the vacuum chamber. The resulting Lorentz force $\mathbf{F_L} = 2\pi R_p(I_p\mathbf{e}_\phi \times B_v\mathbf{e_z})$ should equilibrate the radial force
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- Question: in what direction should the vertical field be oriented?
  - Answer: Use right-hand rule to find the B_v direction giving an inward force $\mathbf{F_L} \sim \mathbf{I} \times \mathbf{B_v}$. For our convention $\text{sign}(B_v) = -\text{sign}(I_p)$.
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- Question: What is the strength of the required vertical magnetic field?
- Answer:

$$B_{v,req} = \frac{-F_L}{2\pi R_p I_p} = -\frac{\mu_0 I_p}{4\pi R_p}\left(\ln\frac{8R_p}{a\sqrt{\kappa}} + \beta_p + \frac{\ell_i}{2} - \frac{3}{2}\right) = -\frac{\mu_0 I_p \Gamma}{4\pi R_p} \tag{2}$$

Radial force balance - sketch of derivation

- MHD (MagnetoHydroDynamics) force balance (Newton's Second Law):

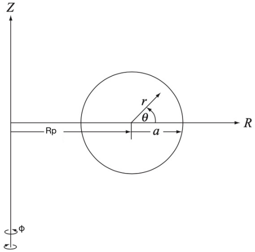
$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{3}$$

- Assume limit $\rho = 0$ 'instantaneous force balance' (on time scales of interest).

$$\mathbf{J} \times \mathbf{B} - \nabla p = 0 \tag{4}$$

Radial force balance - sketch of derivation

- Let's look at magnetic field in this coordinate system
  $R = R_p + r\cos\theta$, $Z = r\sin\theta$:



- A simplified, large-aspect ratio model for pressure and fields in a tokamak is given by [1, Section 11.7]:

$$p = p(r), \quad \mathbf{B} = \underbrace{\frac{R_p}{R} B_\phi(r)\mathbf{e}_\phi + \frac{R_p}{R} B_\theta(r)\mathbf{e}_\theta}_{\text{plasma}} + \underbrace{B_v \mathbf{e}_z}_{\text{external}} \qquad (5)$$

Radial force balance - sketch of derivation

- Radial (or toroidal) force balance is expressed as: $\int \mathbf{e_r} \cdot [\mathbf{J} \times \mathbf{B} - \nabla p] \mathrm{d}\mathbf{r}$
- Using $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$, we get

$$\mathbf{e_r} \cdot (\mathbf{J} \times \mathbf{B}) = -cos\theta \left[\frac{R_p^2}{R^2} \frac{\partial}{\partial r} \left(\frac{B_\phi^2}{2\mu_0} \right) + \frac{R_p}{R} \frac{B_\theta}{\mu_0 r} \frac{\partial}{\partial r} \left(\frac{R_p}{R} r B_\theta \right) \right] - \frac{B_v}{\mu_0 r} \frac{\partial}{\partial r} \left(\frac{R_p}{R} r B_\theta \right) \quad (6)$$

- We see there are 4 contributions to radial force balance. The p term, a B_ϕ term, a B_θ and B_v
- A more intuitive form [2] where the three forces are visible is:

$$F_R = \frac{I_p^2}{2} \frac{\partial}{\partial R_p} (L_e + L_p) + 4\pi^2 \int_0^a \left(p - \frac{B_0(B_\phi(r) - B_0)}{\mu_0} \right) r dr \quad (7)$$

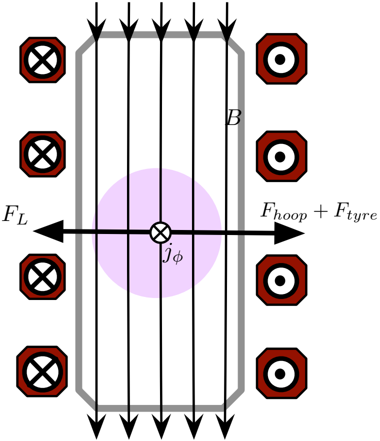
with $L_e(R_p) = \mu_0 R_p (\ln(8R_p/a) - 2)$

Vertical field generation

- Question: Draw a simple set-up of PF coil currents that gives the required field.

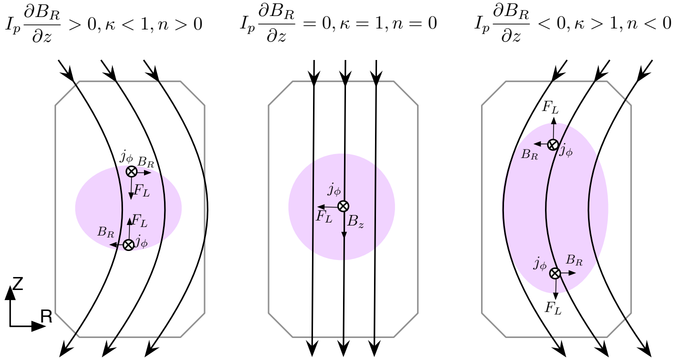
Vertical field generation

- Question: Draw a simple set-up of PF coil currents that gives the required field.
- Answer: many options, example:



Vertical field and plasma elongation

- Adding some curvature to the vertical field can be used to *elongate* the plasma. Curvature index: $n = -\frac{R_0}{B^a_{z0}} \frac{\partial B^a_z}{\partial R}$
- Elongation $\kappa = \frac{Z_{max} - Z_{min}}{R_{max} - R_{min}}$, $\kappa > 1$ is desired, since it is gives a higher confinement.



Some relations regarding the curvature index

- Curvature index definition: curvature of field generated by external coils:

$$n = -\frac{R_0}{B^a_{z0}} \frac{\partial B^a_z}{\partial R} \tag{8}$$

- In the plasma region, the field $\mathbf{B^a_p}$ is irrotational: $\nabla \times \mathbf{B^a_p} = 0$ since the current sources for this field are outside the plasma, so

$$\frac{\partial B^a_Z}{\partial R} = \frac{\partial B^a_R}{\partial Z} \tag{9}$$

- Hence the curvature index can also be written as

$$n = -\frac{R_0}{B^a_{z0}} \frac{\partial B^a_R}{\partial Z} \tag{10}$$

Radial position stability

- Suppose we have the plasma in radial equilibrium, so $B_{z,0} = B_{v,req}$. Is this equilibrium stable?

Radial position stability

- Suppose we have the plasma in radial equilibrium, so $B_{z,0} = B_{v,req}$. Is this equilibrium stable?
- It will be stable if

$$\frac{\partial F_R}{\partial R_p} < 0, \tag{11}$$

where

$$F_R = -2\pi R_p I_p (B_z - B_{v,req}), \tag{12}$$

$B_{v,req}$ is the vertical field required for equilibrium, B_z is the applied vertical field, (2) and R_p is the radial position of the plasma center.

Radial position stability

- Calculating the derivative, assuming at equilibrium $B_z = B^a_{z0} = B_{v,req}$, $R_0 = R_p$:

$$\begin{aligned} &\frac{\partial}{\partial R_p}\left(-2\pi R_p I_p (B_z - B_{v,req})\right) \\ &= -2\pi I_p \underbrace{(B_z - B_{v,req})}_{=0} - 2\pi R_p I_p \left(\frac{\partial B_z}{\partial R} - \frac{\partial B_{v,req}}{\partial R_p}\right) \\ &= 2\pi R_p I_p \left(-\frac{n B_z}{R_p} + \frac{\partial}{\partial R_p}\frac{\mu_0 I_p \Gamma(R_p)}{4\pi R_p}\right) \\ &= \frac{R_p I_p \mu_0}{2}\left\{\frac{n I_p \Gamma}{R_p^2} + \Gamma \frac{1}{R_p}\frac{\partial I_p}{\partial R_p} + (1-\Gamma)\frac{I_p}{R_p^2}\right\} \end{aligned} \quad (13)$$

- To calculate $\frac{\partial I_p}{\partial R_p}$ we assume the magnetic flux contained by the plasma ring stays constant while varying R_p: $\frac{\partial}{\partial R_p}\left[\pi R_p^2 B_{V,req} - L_e I_p\right] = 0$. where L_e is the external inductance of the plasma.

Radial position stability

- Approximating $L_e = \mu_0 R_p \left[\ln \frac{8R_p}{a} - 2\right]$ (analytic equation for a ring), also assuming $\ln(8R_p/a) \gg 1$, this yields

$$\frac{I_p}{2} - R_p \frac{\partial I_p}{\partial R_p} - I_p = 0 \rightarrow \frac{\partial I_p}{\partial R_p} = -\frac{I_p}{2R_p} \quad (14)$$

- With this, our condition becomes

$$\frac{\partial F_R}{\partial R_p} = \frac{I_p^2 \mu_0 \Gamma}{2R_p} \left\{ n - 1 - \frac{1}{2} \right\} < 0 \quad (15)$$

  and we get the stability condition $n < \frac{3}{2}$
- Now recall that for elongated plasmas, $n < 0$. So all plasmas of practical interest, including circular ones (n=0), are *radially stable*.

Vertical equilibrium & stability

- The plasma will be in equilibrium at the location where $B_r = 0$. At this location there is no net vertical force.
- The stability criterion is much easier to calculate, $\frac{\partial F_z}{\partial Z_0}<0$, with $F_z = -2\pi I_p B_R$ and Z_0 the Z position of the plasma, becomes

$$\frac{\partial B_R}{\partial Z_0} > 0 \tag{16}$$

- Using the fact that $\frac{\partial B_R^a}{\partial Z} = \frac{\partial B_Z^a}{\partial R}$, plus the definition of $n = -\frac{R_p}{B_{z0}^a}\frac{\partial B_z^a}{\partial R}$ and recalling $B_{Z0}^a < 0$ for $I_p > 0$, this yields the condition (for stability).

$$n > 0 \tag{17}$$

- Note that for elongated plasmas we need $n < 0$. So for high-performance, elongated tokamak plasmas will be vertically unstable!

Subsection 2

Rigid plasma position stability and control

Radial position control

- Radial position dynamics:

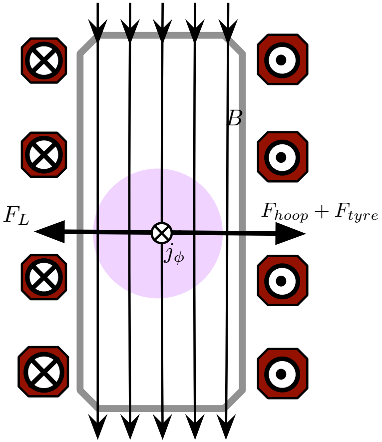
$$m_p \frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = F_{R,Lorentz} + F_{R,hoop} + F_{R,tyre} + F_{R,1/R} \quad (18)$$

$$= \frac{\mu_0 I_p^2}{2} \Gamma(R, \beta_p, \ell_i) + 2\pi R I_p B_Z(R, Z, \mathbf{I_a}, \mathbf{I_v}) \quad (19)$$

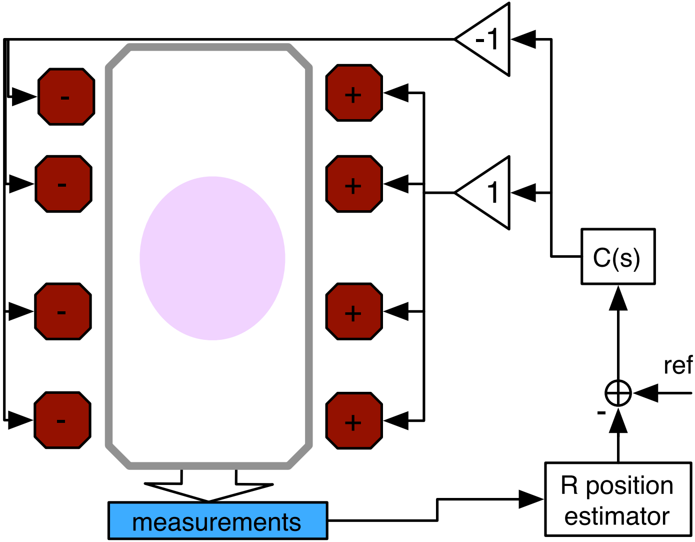
where, from (1), $\Gamma = \left(\ln \frac{8R}{a\sqrt{\kappa}} + \beta_p + \frac{\ell_i}{2} - \frac{3}{2} \right)$

- Though it is stable, the radial position equilibrium is affected by I_p, β, ℓ_i. Changes in these quantities will change the radial position.
- Radial position is controlled by varying B_Z in feedback, via changes in I_a.

Radial position control



Radial position control



Vertical position stability

- Vertical force balance:

$$m_p \frac{\mathrm{d}^2 Z}{\mathrm{d}t^2} = F_{Z,Lorentz} \tag{20}$$

$$= -2\pi R I_p B_R(R_p, Z_p, \mathbf{I_a}, \mathbf{I_v}) \tag{21}$$

Separate radial field contribution from coils and from vessel :

$$B_R = B_R^a + B_R^v = B_R^a - \frac{1}{2\pi R_p} \frac{\partial M_{pv}}{\partial Z_p} I_v \tag{22}$$

Assume small displacement $z = \delta Z_p$, small vessel current reaction δI_v and fixed I_a. Linearizing the equation[1]:

$$m_p \ddot{z} = -2\pi R_p I_{p0} \left(\frac{\partial B_R^a}{\partial Z_p} z - \frac{1}{2\pi R_p} \frac{\partial M_{pv}}{\partial Z_p} \delta I_v \right) \tag{23}$$

[1] NB: $B_R = 0$ at equilibrium position

Vertical position stability

$$m_p\ddot{z} + 2\pi R_p I_{p0}\left(\frac{\partial B_R^a}{\partial Z_p} z - \frac{1}{2\pi R_p}\frac{\partial M_{pv}}{\partial Z_p}\delta I_v\right) = 0 \tag{24}$$

- For the part of the field generated by the PF coils: $\nabla \times \mathbf{B^a} = 0$ so that $\frac{\partial B_R^a}{\partial Z} = \frac{\partial B_Z^a}{\partial R}$.
- We then use the decay index of the vertical field along R:

$$n = -\frac{R_0}{B_{z0}^a}\frac{\partial B_z^a}{\partial R_0} \tag{25}$$

and write:

$$\frac{\partial B_R^a}{\partial Z_p} = -\frac{nB_{z0}}{R_p} = \frac{\mu_0 I_{p0}\Gamma}{4\pi R_p^2} n \tag{26}$$

So

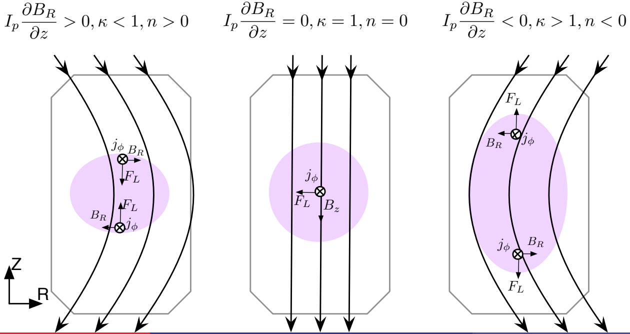
$$m_p\ddot{z} + \frac{\mu_0 \Gamma I_{p0}^2 n}{2R_p} z - I_{p0}\frac{\partial M_{pv}}{\partial Z}\delta I_v = 0 \tag{27}$$

Vertical position stability

- Define $\omega_1^2 = \frac{\mu_0 I_p^2 \Gamma}{2 m_p R_p}$

$$\ddot{z} + \omega_1^2 n z - \frac{I_{p0}}{m_p}\frac{\partial M_{pv}}{\partial Z_p}\delta I_v = 0 \tag{28}$$

- If there is no vacuum vessel, $I_v = 0$, the poles of $\ddot{z} + \omega_1^2 n z = 0$ are $s = \pm\sqrt{-n}\omega_1$. For $n < 0$ ($\kappa > 1$) there will be one **unstable** value.



Vertical position stability - vessel currents

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- Let the vacuum vessel mode be modeled by one up-down asymmetric eigenmode $\delta \mathbf{I_v}(t) = \mathbf{v_e} I_e(t)$ with $I_e(t)$ the (scalar) current in the relevant eigenmode, equivalent self/resistivity L_e, R_e and inductive coupling to plasma M_{ep}.

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- Linearized circuit equation for that vessel 'mode':

$$L_e \dot{I}_e + R_e I_e + \frac{\mathrm{d}(M_{ep} I_p)}{\mathrm{d}t} = 0 \tag{29}$$

$$L_e \dot{I}_e + R_e I_e + I_{p0} \frac{\partial M_{ep}}{\partial Z_p} \dot{z} = 0 \tag{30}$$

Vertical position stability - vessel currents

- We can thus write in the Laplace domain:

$$\left(s^2+\omega_1^2 n\right) z - \frac{I_{p0}}{m_p}\frac{\partial M_{pe}}{\partial Z_p} I_e = 0 \quad \text{plasma vertical position} \qquad (31)$$

$$\left(s+\frac{R_e}{L_e}\right) I_e + s\frac{I_{p0}}{L_e}\frac{\partial M_{ep}}{\partial Z_p} z = 0 \quad \text{vessel response} \qquad (32)$$

Defining $n_c = \frac{2R_p}{\mu_0 \Gamma L_e}\left(\frac{\partial M_{ep}}{\partial Z_p}\right)^2$, this system has the characteristic equation:

$$\left(s^2+\omega_1^2 n\right)\left(s+\frac{R_e}{L_e}\right) + s\omega_1^2 n_c = 0 \qquad (33)$$

$$= s^3 + s^2\frac{R_e}{L_e} + \omega_1^2\left((n+n_c)s + \frac{R_e}{L_e}n\right) = 0 \qquad (34)$$

Vertical position stability - stability analysis

- In the limit $m_p = 0$ and $\omega_1 >> R_e/L_e$ we may neglect 3rd and 2nd order terms. We get, after some algebra:

$$z = \frac{I_{p,0}}{m_p\omega_1^2 n}\frac{\partial M_{ep}}{\partial Z_p} I_e = \frac{2R_p}{\mu_0 I_{p,0}\Gamma n}\frac{\partial M_{ep}}{\partial Z_p} I_e \tag{35}$$

Substituting this in the vessel equation:

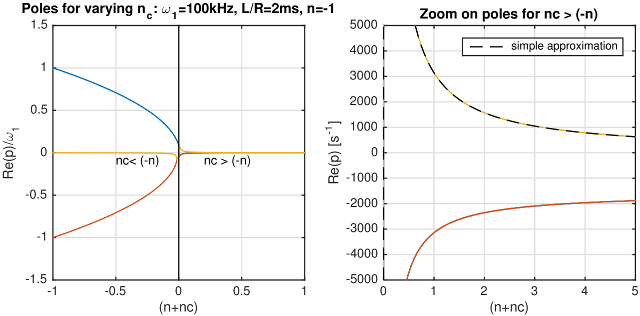
$$\left(1 + \frac{n_c}{n}\right)\dot{I}_e + \frac{R_e}{L_e} I_e = 0 \tag{36}$$

with $n_c = \frac{2R_p}{\mu_0\Gamma L_e}\left(\frac{\partial M_{ep}}{\partial Z_p}\right)^2$. This has a pole:

$$s = \frac{-R_e/L_e}{1 + n_c/n} \tag{37}$$

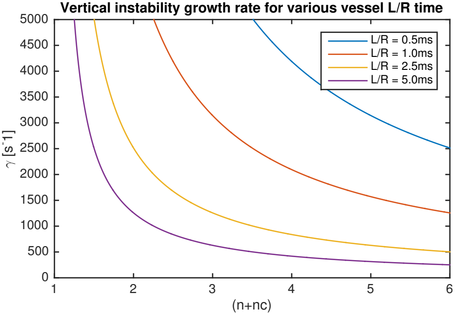
This is valid for $(n + n_c) > 0$ or $n_c/n < -1$. Recall $n < 0$ for elongated plasmas.

Vertical position stability - stability analysis



- We can distinguish two regions
  - For $n_c < -n$ the fastest unstable eigenmode is of order ω_1. In this case the vessel L/R time is too slow, and eddy currents appear too slowly to have an effect on the mode - no hope for stabilization.
  - For $n_c > -n$ the vessel generates sufficient eddy currents to slow down the instability. However the current dissipates with $\tau \sim L/R$: we must still stabilize using external coils.

Vertical position control



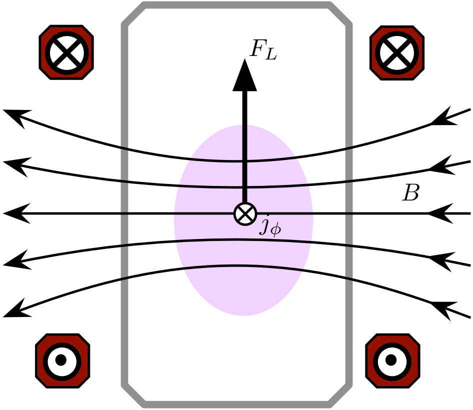
- Good for stability (low growth rate):
  - Large L_e/R_e Low vessel resistivity R_e or large L_e.
  - Large n_c (large $\frac{\partial M_{pe}}{\partial Z_p}$, or small Γ: low ℓ_i, low β_p)
  - Lower κ less negative n ($n > 0$ is always stable)
- Vertical control is required for all elongated tokamaks, hence well-studied. Early paper: [5], see also [6] [3], .
- Critical for ITER: large superconducting coils far away from the plasma. In-vessel coils will help. [7]

Vertical position control

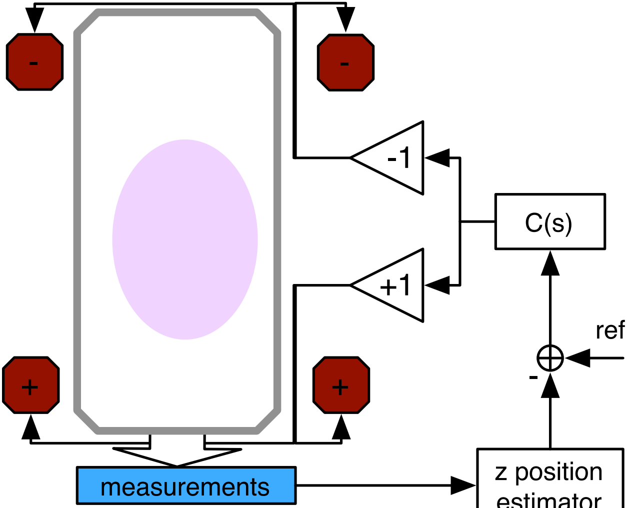
- Question: draw a set of PF coils and required direction of PF coil current change that would generate the required radial field

Vertical position control

- Question: draw a set of PF coils and required direction of PF coil current change that would generate the required radial field
- Answer: top - bottom antisymmetric current change will do.

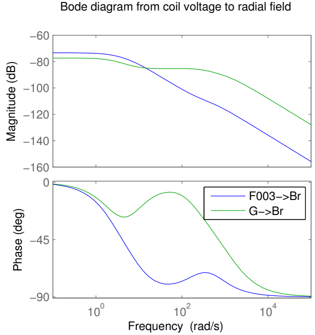


Vertical position control



Vertical position control - in-vessel coils

- Coils outside the vessel have more difficulty penetrating the vessel at high frequencies.
- Therefore TCV, as many other tokamaks, has *in-vessel* coil



- Another option, chosen by ASDEX-Upgrade tokamak, is to install in-vessel *passive coils* which reduce the growth rate.

Vertical position control model

- Let's include the circuit equation for an 'active' coil:

$$z - \frac{2R_p}{\mu_0 I_{p,0}\Gamma n}\frac{\partial M_{pe}}{\partial Z_p} I_e - \frac{2R_p}{\mu_0 I_{p,0}\Gamma n}\frac{\partial M_{pa}}{\partial Z_p} I_a = 0 \quad \text{position} \tag{38}$$

$$L_e \dot{I}_e + R_e I_e + I_{p0}\frac{\partial M_{ep}}{\partial Z_p}\dot{z} + M_{ea}\dot{I}_a = 0 \quad \text{vessel} \tag{39}$$

$$L_a \dot{I}_a + R_a I_a + I_{p0}\frac{\partial M_{ap}}{\partial Z_p}\dot{z} + M_{ae}\dot{I}_e = V_a \quad \text{coil} \tag{40}$$

Vertical position control model

- Substituting the static relation (38) into the two circuit equations and taking the Laplace transform:

$$L_e\left(1+\frac{n_e}{n}\right)sI_e + R_e I_e + M_{ea}\left(1+\frac{n_{ea}}{n}\right)sI_a = 0 \qquad (41)$$

$$L_a\left(1+\frac{n_a}{n}\right)sI_a + R_a I_a + M_{ea}\left(1+\frac{n_{ea}}{n}\right)sI_e = V_a \qquad (42)$$

- Where: $n_e = \frac{2R_p}{\mu_0 \Gamma L_e}\left(\frac{\partial M_{ep}}{\partial Z_p}\right)^2$, $n_a = \frac{2R_p}{\mu_0 \Gamma L_a}\left(\frac{\partial M_{ap}}{\partial Z_p}\right)^2$, $n_{ea} = \frac{2R_p}{\mu_0 \Gamma M_{ea}}\left(\frac{\partial M_{ap}}{\partial Z_p}\right)\left(\frac{\partial M_{ep}}{\partial Z_p}\right)$

Vertical position control model

- We rewrite the equation in matrix form as

$$M\dot{I} + RI = bV \tag{43}$$

- with $I = [I_e, I_a]^T$, $M = \begin{bmatrix} L_e(1+\frac{n_e}{n}) & M_{ea}(1+\frac{n_{ea}}{n}) \\ M_{ea}(1+\frac{n_{ea}}{n}) & L_a(1+\frac{n_a}{n}) \end{bmatrix}$
$R = \begin{bmatrix} R_e & 0 \\ 0 & R_a \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Furthermore, assuming we measure z and the coil current I_a, the measurement equation becomes $\begin{bmatrix} z \\ I_a \end{bmatrix} = \begin{bmatrix} C_z \\ S_{Ia} \end{bmatrix} I$ with
$C_z = \frac{2R_p}{\mu_0 I_{p0} \Gamma n} \begin{bmatrix} \frac{\partial M_{pe}}{\partial Z_p} & \frac{\partial M_{pa}}{\partial Z_p} \end{bmatrix}$ and $S_{I_a} = [0\ ,\ 1]$

(44)

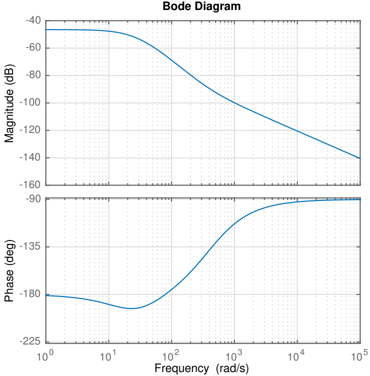
Vertical position control model

- Numerical example:

```
%% Parameters for TCV-like case
Ra = 4*0.0391; La = 4*0.0074;
Re = 5.5657e-5; Le = 4.5158e-7; Mea = 4.74e-5;
n = -0.5; na = 0.421; ne = 1.027; nea = 1.47;
Ip0 = 200e3; R0 = 0.88; mu0=4e-7*pi; Gamma = 2.5;
dMpadz = 1.32e-4; dMpedz = 8.7931e-7;

%% Model
M = [Le*(1+ne/n) Mea*(1+nea/n)
  Mea*(1+nea/n) La*(1+na/n)];
R = diag([Re,Ra]); b = [0;1];
Cz = 2*R0/(mu0*Ip0*Gamma*n)*[dMpedz,dMpadz];

A = -M\R; B=M\b; C = [Cz;0 1];
sys = ss(A,B,C,0); sysZ = sys(1,:);

%% Bode plot
clf; figure(1); set(gcf,'position',[0 0 400 400])
Wn = logspace(0,5,101);
bode(sysZ,Wn); grid on;
```

Vertical position control model

- Bode diagram of open-loop $z(s)/V_a(s)$



- Question: Can we use concepts of gain margin and phase margin Bode plot to design a stabilizing controller?
- Answer: No! Bode stability margins apply only to open-loop *stable* systems!

The Nyquist Criterion - Summary

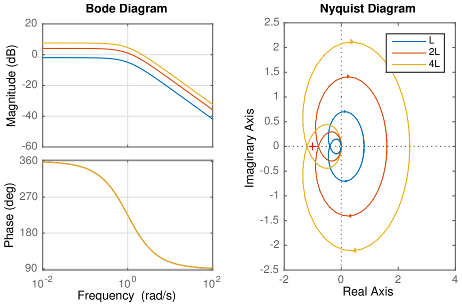
- Nyquist plot: trajectory of *loop gain* $L(j\omega)$ for $\omega \in \mathbb{R}$
- It holds that $N = Z - P$ where:
  - N: Number of *clockwise encirclements* minus the number of *counter-clockwise encirclements* by $L(j\omega)$ of $(-1 + 0j)$.
  - P: Number of RHP (unstable) poles of the open-loop $L(s)$.
  - Z: Number of RHP zeros of (1+L) (unstable poles of the *closed-loop* system)
- For a stable closed-loop, we desire $Z = 0$.
- For open-loop *stable* systems: $P = 0$, $N = Z$: so we use the Nyquist plot to check that we avoid encircling the -1 point.
- For open-loop *unstable* systems: $P > 0$, so $N < 0$ we use the Nyquist plot to check that we encircle the -1 point *counterclockwise* P times.

Nyquist criterion: example

- Open-loop system:

$$P(s) = \frac{0.8(s-1)}{(s+1)^2} \tag{45}$$

- Controller: $K(s) = K_p$. Open-loop: $L(s) = K_p P(s)$.
- Nyquist contours for $K_p = \{1, 2, 4\}$: when does the closed-loop become unstable?

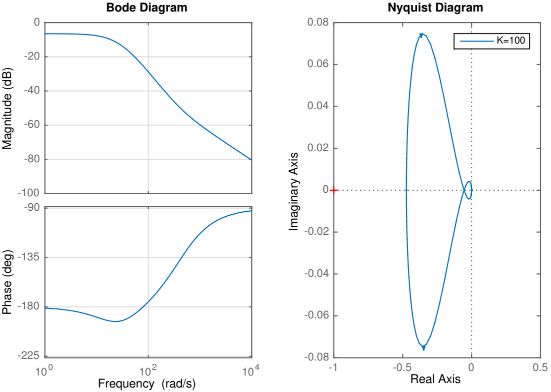


Nyquist analysis for vertical control

- Open-loop system: $P(s) = \frac{0.0094563(s+407.5)}{(s-38.54)(s+21.14)}$
- Controller: $K(s) = K_p$. Open-loop: $L(s) = K_p P(s)$.
- Nyquist contours for $K_p = \{10, 1000, 5000\}$: when does the closed-loop become stable?

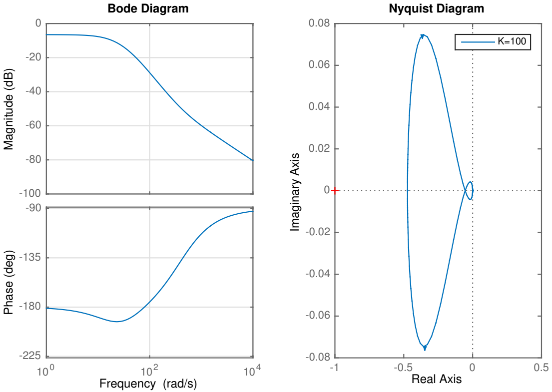
Nyquist analysis for vertical control

- $K_p = 100$



Nyquist analysis for vertical control

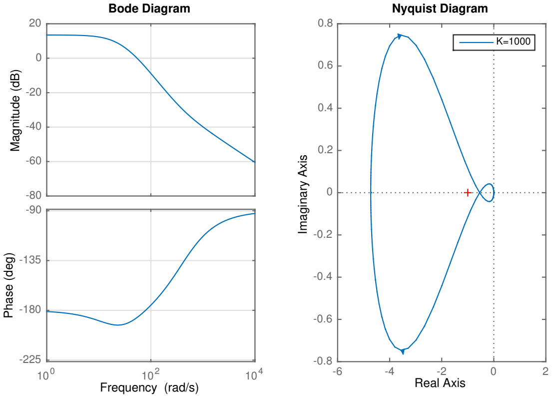
- $K_p = 100$



- Unstable: no encirclement

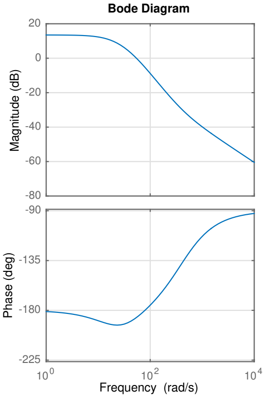
Nyquist analysis for vertical control

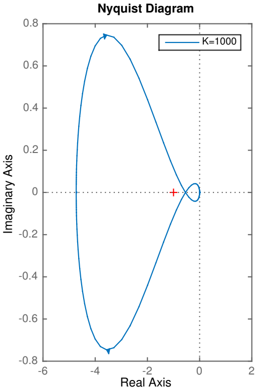
- $K_p = 1000$



Nyquist analysis for vertical control

- $K_p = 1000$

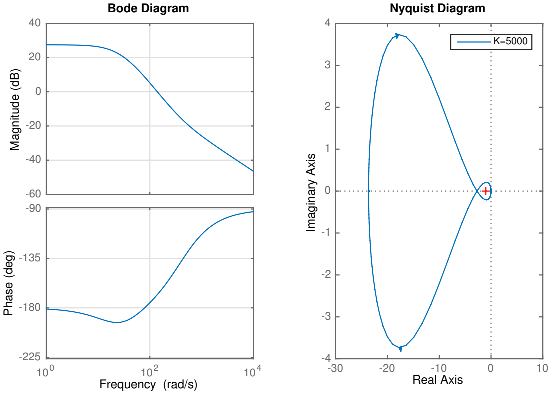




- Unstable: 1 clockwise encirclement

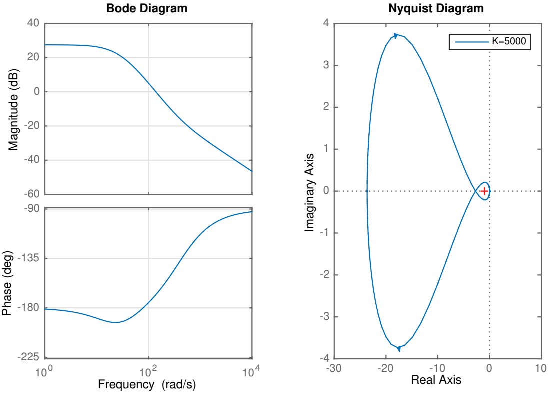
Nyquist analysis for vertical control

- $K_p = 5000$



Nyquist analysis for vertical control

- $K_p = 5000$



- Stable: 1 counter-clockwise encirclement

Subsection 3

The RZIp model

RZIp model - building blocks

- The RZIp model combines:
  - Circuit equations for active and passive conductors
  - Circuit equation for plasma
  - Radial and vertical force balance for plasma position
- It is a *linearized* model around a time-invariant equilibrium position and current R_0, Z_0, I_{p0}, the resulting model is *linear*.
- Note that to sustain this time-invariant plasma equilibrium, we may have a time-varying $\mathbf{I_a^0}(t), \mathbf{I_u^0}(t)$.
- Procedure for derivation
  - Take force balance equations setting $m_p = 0$: assume instantaneous force balance, neglecting inertia.
  - Linearize force balance around equilibrium state.
  - Substitute linearized force balance into circuit equations.
- It still treats the plasma like a rigid body, but it is a reasonably good model for designing and testing controllers axisymmetric control of plasma current and position.
- For details, see [8]

Generalized rigid plasma evolution model

- Let's collect all the circuit and passive currents into

$$I_e = \begin{bmatrix} \mathbf{I_a} \\ \mathbf{I_u} \end{bmatrix} \tag{46}$$

- Write the circuit equation with a generic induction term due to the plasma:

$$M_{ee}\dot{I}_e + R_{ee}I_e + \dot{\psi}_{ep} = V_e \tag{47}$$

with $\psi_{ep} = \frac{d}{dt}(M_{ey}I_y) = M_{ey}\frac{d}{dt}(I_y)$. This expression works for any time-varying change of plasma current, not only rigid ones.
- Now we parametrize the plasma current distribution using the rigid body assumption as [2] $I_y = I_y(R_p, Z_p, I_p)$
- Using the chain rule, we get

$$\frac{dI_y}{dt} = \frac{\partial I_y}{\partial R_p}\dot{R}_p + \frac{\partial I_y}{\partial Z_p}\dot{Z}_p + \frac{\partial I_y}{\partial I_p}\dot{I}_p \tag{48}$$

[2] Previously we used I_x and M_{xx} on the x grid, now we use I_y, M_{yy} etc on a different y grid for technical reasons.

Generalized rigid plasma evolution model

- We also recall the Radial and Vertical Force balance, which are of the form

$$F_R(R_p, Z_p, I_p, I_e) = 0; \quad (49)$$

$$F_Z(R_p, Z_p, I_p, I_e) = 0; \quad (50)$$

Linearizing these, we get

$$\frac{\partial F_R}{\partial R_p}\delta R_p + \frac{\partial F_R}{\partial Z_p}\delta Z_p + \frac{\partial F_R}{\partial I_p}\delta I_p + \frac{\partial F_R}{\partial I_e}\delta I_e = 0; \quad (51)$$

$$\frac{\partial F_Z}{\partial R_p}\delta R_p + \frac{\partial F_Z}{\partial Z_p}\delta Z_p + \frac{\partial F_Z}{\partial I_p}\delta I_p + \frac{\partial F_Z}{\partial I_e}\delta I_e = 0; \quad (52)$$

or:

$$\begin{pmatrix} \frac{\partial F_R}{\partial R_p} & \frac{\partial F_R}{\partial Z_p} \\ \frac{\partial F_Z}{\partial R_p} & \frac{\partial F_Z}{\partial Z_p} \end{pmatrix} \begin{bmatrix} \delta R_p \\ \delta Z_p \end{bmatrix} + \begin{pmatrix} \frac{\partial F_R}{\partial I_p} & \frac{\partial F_R}{\partial I_e} \\ \frac{\partial F_Z}{\partial I_p} & \frac{\partial F_Z}{\partial I_e} \end{pmatrix} \begin{bmatrix} \delta I_p \\ \delta I_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (53)$$

Generalized rigid plasma evolution model

- Inverting and rearranging:

$$\begin{bmatrix} \delta R_p \\ \delta Z_p \end{bmatrix} = \begin{bmatrix} \frac{\partial R_p}{\partial I_p} \\ \frac{\partial Z_p}{\partial I_p} \end{bmatrix} \delta I_p - \begin{bmatrix} \frac{\partial R_p}{\partial I_e} \\ \frac{\partial Z_p}{\partial I_e} \end{bmatrix} \delta I_e \qquad (54)$$

with

$$\begin{bmatrix} \frac{\partial R_p}{\partial I_p} \\ \frac{\partial Z_p}{\partial I_p} \end{bmatrix} = \begin{pmatrix} \frac{\partial F_R}{\partial R_p} & \frac{\partial F_R}{\partial Z_p} \\ \frac{\partial F_Z}{\partial R_p} & \frac{\partial F_Z}{\partial Z_p} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_R}{\partial I_p} \\ \frac{\partial F_Z}{\partial I_p} \end{pmatrix}, \quad \begin{bmatrix} \frac{\partial R_p}{\partial I_e} \\ \frac{\partial Z_p}{\partial I_e} \end{bmatrix} = \begin{pmatrix} \frac{\partial F_R}{\partial R_p} & \frac{\partial F_R}{\partial Z_p} \\ \frac{\partial F_Z}{\partial R_p} & \frac{\partial F_Z}{\partial Z_p} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_R}{\partial I_e} \\ \frac{\partial F_Z}{\partial I_e} \end{pmatrix}$$

- Now take the time derivative and combine this with circuit equation (47) and the expression for $\frac{dI_y}{dt}$ of (48)

$$M_{ee}\dot{I}_e + M_{ey} \begin{bmatrix} \frac{\partial I_y}{\partial R_p} & \frac{\partial I_y}{\partial Z_p} \end{bmatrix} \left(\begin{bmatrix} \frac{\partial R_p}{\partial I_p} \\ \frac{\partial Z_p}{\partial I_p} \end{bmatrix} \delta \dot{I}_p - \begin{bmatrix} \frac{\partial R_p}{\partial I_e} \\ \frac{\partial Z_p}{\partial I_e} \end{bmatrix} \delta \dot{I}_e \right) + \frac{\partial I_y}{\partial I_p} \dot{I}_p + R_{ee} I_e = V_e \qquad (55)$$

Transformer currents

- To sustain the plasma equilibrium inductively, a time-varying external current trajectory $I_{e0}(t)$ may be required.
- A key step in the derivation is to realize that because we assume $\dot{R}_{p0} = \dot{Z}_{p0} = \dot{I}_{p0} = 0$, this implies $\delta\dot{I}_p = \dot{I}_p$ (trivially) and also: $\frac{\partial R_p}{\partial I_e}\dot{I}_{e0} = \frac{\partial Z_p}{\partial I_{e0}}\dot{I}_{e0} = 0$.
- This means that $\dot{I}_{e0}$: (typically the time-varying currents in the 'Ohmic coils') does not change the force balance:
- $\frac{\partial F_R}{\partial I_e}\dot{I}_{e0} = 0$, $\frac{\partial F_Z}{\partial I_e}\dot{I}_{e0} = 0$
- This implies the magnetic field stays constant while coils evolve in direction $\dot{I}_{e0}$, From (38) we see $\frac{\partial M_{pe}}{\partial Z_p}\dot{I}_{e0} = 0$, $\frac{\partial M_{pe}}{\partial R_p}\dot{I}_{e0} = 0$

Transformer currents

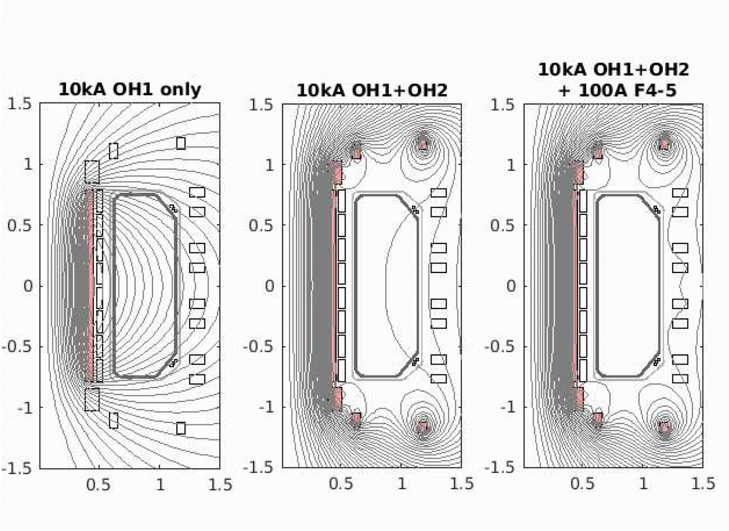


Figure: TCV flux pattern for various combinations of coil currents

Generalized rigid plasma evolution model

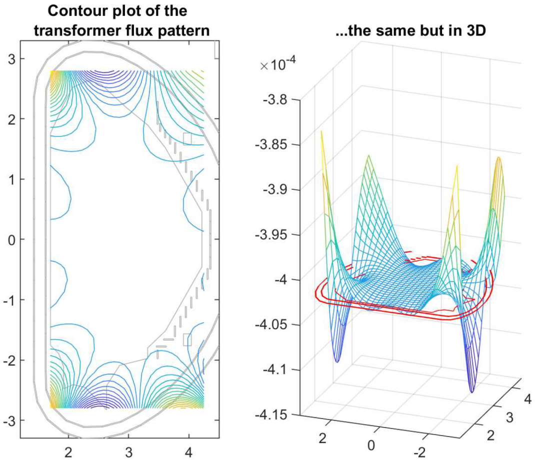


Figure: EAST flux pattern for I_p control, from combination of PF coils [from A. Mele lecture slides]

Generalized rigid plasma evolution model

- Now since $\frac{\partial R_p}{\partial I_e}(\dot{I}_{e0}(t)) = 0$, $\frac{\partial R_p}{\partial I_e}\delta\dot{I}_e = \frac{\partial R_p}{\partial I_e}(\dot{I}_{e0}(t) + \delta\dot{I}_e) = \frac{\partial R_p}{\partial I_e}\dot{I}_e$ (and similar for Z_p)
- Collecting terms yields:

$$(M_{ee} + X_{ee})\dot{I}_e + (M_{ep} + X_{ep})\dot{I}_p + R_{ee}I_e = V_e \tag{56}$$

where:

- $X_{ee} = M_{ey}\left(\frac{\partial I_y}{\partial R_p}\frac{\partial R_p}{\partial I_e} + \frac{\partial I_y}{\partial Z_p}\frac{\partial Z_p}{\partial I_e}\right)$
- $X_{ep} = M_{ey}\left(\frac{\partial I_y}{\partial R_p}\frac{\partial R_p}{\partial I_p} + \frac{\partial I_y}{\partial Z_p}\frac{\partial Z_p}{\partial I_p}\right)$
- $M_{ep} = M_{ey}\frac{\partial I_y}{\partial I_p} = M_{ey}\frac{I_{y0}}{I_{p0}}$

Generalized rigid plasma evolution model

- Now consider the circuit equation for the plasma current:

$$\dot{\psi}_y + R_{yy} I_y = 0 \tag{57}$$

with

$$\dot{\psi}_y = M_{yy}\dot{I}_y + M_{ye}\dot{I}_e \tag{58}$$

- Again parametrizing $I_y = I_y(R_p, Z_p, I_p)$, linearizing as in (48), multiplying from the left by I_{y0}^T/I_{p0}, and assuming plasma resistance does not change with plasma position, yields:

$$\underbrace{\frac{I_{y0}^T}{I_{p0}} M_{yy} \frac{\partial I_y}{\partial I_p}}_{L_{pp}=I_{y0}^T M_{yy} I_{y0}/I_{p0}^2} \dot{I}_p + \underbrace{\frac{I_{y0}^T}{I_{p0}} M_{yy} \frac{\partial I_y}{\partial R_p}}_{M_{pR}=\frac{1}{2}\frac{\partial L_{pp}}{\partial R_p} I_{p0}} \dot{R}_p + \underbrace{\frac{I_{y0}^T}{I_{p0}} M_{yy} \frac{\partial I_y}{\partial Z_p}}_{M_{pZ}=\frac{1}{2}\frac{\partial L_{pp}}{\partial Z_p} I_{p0}} \dot{Z}_p + \underbrace{\frac{I_{y0}^T}{I_{p0}} M_{ye}}_{=M_{pe}} \dot{I}_e + \underbrace{\frac{I_{y0}^T}{I_{p0}} R_{yy} \frac{I_{y0}}{I_{p0}}}_{R_{pp}} I_p = 0 \tag{59}$$

- Hence

$$L_{pp}\dot{I}_p + M_{pR}\dot{R}_p + M_{pZ}\dot{Z}_p + M_{pe}\dot{I}_e + R_{pp} I_p = 0 \tag{60}$$

Generalized rigid plasma evolution model

- Again using (54):

$$L_{pp}\dot{I}_p + \left[\begin{array}{cc} M_{pR} & M_{pZ} \end{array}\right]\left(\left[\begin{array}{c} \frac{\partial R_p}{\partial I_p} \\ \frac{\partial Z_p}{\partial I_p} \end{array}\right]\delta\dot{I}_p - \left[\begin{array}{c} \frac{\partial R_p}{\partial I_e} \\ \frac{\partial Z_p}{\partial I_e} \end{array}\right]\delta\dot{I}_e\right) + M_{pe}\dot{I}_e + R_{pp}I_p = 0 \quad (61)$$

- or:

$$(L_{pp} + X_{pp})\dot{I}_p + (M_{pe} + X_{pe})\dot{I}_e + R_{pp}I_p = 0 \quad (62)$$

with
  - $X_{pp} = \frac{I_{p0}}{2}\left(\frac{\partial L_p}{\partial R_p}\frac{\partial R_p}{\partial I_p} + \frac{\partial L_p}{\partial Z_p}\frac{\partial Z_p}{\partial I_p}\right)$
  - $X_{pe} = \frac{I_{y0}^T}{I_{p0}}M_{yy}\left(\frac{\partial I_y}{\partial R_p}\frac{\partial R_p}{\partial I_e} + \frac{\partial I_y}{\partial Z_p}\frac{\partial Z_p}{\partial I_e}\right)$
  - $M_{pe} = \frac{I_{y0}^T}{I_{p0}}M_{ye}$

Generalized rigid plasma evolution model

- Finally we obtain the complete linearized dynamic model or a rigid plasma.
- Works around an equilibrium plasma current distribution with known R_p, Z_p, I_p

$$\begin{pmatrix} (\mathbf{M_{ee}} + \mathbf{X_{ee}}) & (\mathbf{M_{ep}} + \mathbf{X_{ep}}) \\ (\mathbf{M_{pe}} + \mathbf{X_{pe}}) & (L_{pp} + X_{pp}) \end{pmatrix} \begin{pmatrix} \dot{\mathbf{I}}_{\mathbf{e}} \\ \dot{I}_p \end{pmatrix} \tag{63}$$

$$+ \begin{pmatrix} \mathbf{R_{ee}} & 0 \\ 0 & R_{pp} \end{pmatrix} \begin{pmatrix} \mathbf{I_e} \\ I_p \end{pmatrix} = \begin{pmatrix} \mathbf{V_a} \\ 0 \end{pmatrix} \tag{64}$$

- For elongated plasmas, this has one unstable eigenvalue
- Removing the X_{**} terms yields the model excluding the effects due to the R_p,Z_p motion
- We can combine this with a measurement equation as shown in previous slides.

Subsection 4

Magnetic control overview: TCV example

Controllers for R, Z, I_p

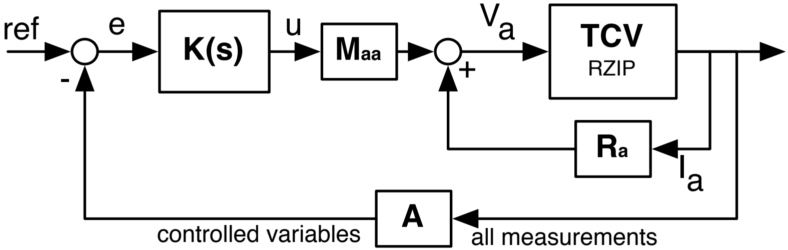
- PF coil direction chosen to control R: $\mathbf{I_R}$
- PF coil direction chosen to control Z: $\mathbf{I_Z}$
- PF coil direction corresponding to OH coils $\mathbf{I_{oh}}$
- Controller for all three quantities:

$$\mathbf{V} = \mathbf{M_{aa}} \mathbf{u} + \mathbf{R} \mathbf{I_a} \tag{65}$$

with

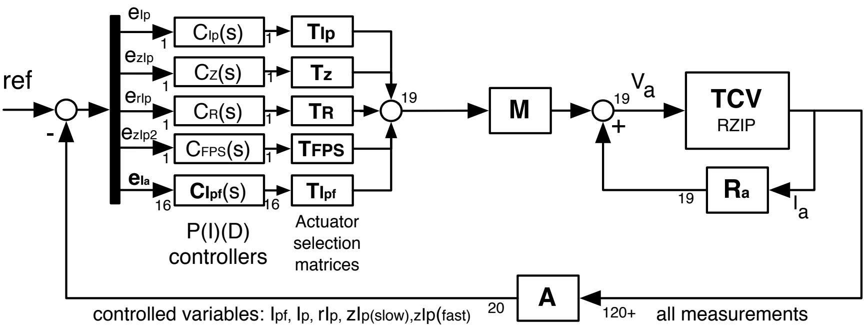
$$\mathbf{u} = \begin{pmatrix} \mathbf{I_R} & \mathbf{I_Z} & \mathbf{I_{oh}} \end{pmatrix} \begin{pmatrix} K_R(s)((RI_p)_{ref} - RI_p) \\ K_Z(s)\,((ZI_p)_{ref} - ZI_p) \\ K_{I_p}(s)(I_p - I_{p,ref}) \end{pmatrix} \tag{66}$$

TCV magnetic control overview



- R_a: resistive coil voltage compensation
- M: mutual decoupling of PF coils
- A: controlled variables as linear combinations of magnetic measurements.
- $K(s)$: linear MIMO controller

TCV magnetic controllers



- C_{Ip}: P controller. T_{I_p} selects OH1, OH2
- C_Z: PD controller. T_z selects F coil combination for Z control
- C_R: P controller. T_r selects F coil combination for R control
- C_{FPS} D controller. T_{FPS} selects internal G coil only
- $C_{I_{PF}}$ P controller. $T_{I_{PF}} \perp \mathrm{Im}[T_z, T_r]$
- C_{DOH}: P controller. T_{DOH} controls $I_{OH1} - I_{OH2}$ to 0 [not shown]

TCV magnetic controllers - details

- Why two vertical position observers: 1 is accurate but noisy, 2 is less accurate but less noisy
- Write combined controller $K(s) = TC(s)$ with:
  - $T = [T_{Ip}, T_z, T_r, T_{I_{PF}}, T_{FPS}]$ and $C(s) = \mathrm{diag}[C_{Ip}, C_Z, C_R, C_{I_{PF}}, C_{FPS}]$.
  - Note that columns of T are orthogonal by design. This means the controller action is *decoupled* in the 19-dimensional space of (16x)PF + (2x)OH + (1x)G coil actuators.
- Final control law: $\mathbf{V_a} = \mathbf{M_{aa}TC(s)(r - Ay) + R_aS_ay + V_{ff}}$
  - S_a: select I_a from measurements y.
  - r references: zero for rIp, zIp errors, $I_{PF,ref}$, I_p from feedforward or shape controller (next class)
- Many practical details not discussed, e.g. implementation, feedforwards..

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