Magnetic modeling and control of tokamaks, Part II: Plasma current control and magnetic measurements

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EPFL Doctoral School Course PHYS-748, February 2025

EPFL

Outline I

1. Modeling and control of the plasma current

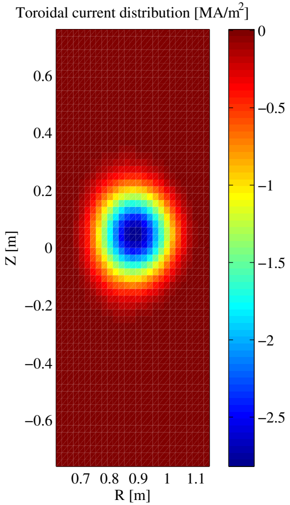
2. Magnetic measurements
   - Estimating plasma position from measurements

Section 1

Modeling and control of the plasma current

Mutual inductance between plasma and conductors

- We start by modeling the plasma as a fixed toroidal conductor carrying a current I_p.
- This current is distributed according to a known distribution $j_\phi(r, z)$
- Let j be modeled with a current distribution vector $\mathbf{I_x}$ of currents [A] on an (r, z) grid, where the sum of the elements of $\mathbf{I_x}$ is I_p.
- Let us also define the matrix $\mathbf{M_{xx}}$ containing the mutual inductance between all the points on this (r, z) grid.



Mutual inductance plasma - conductors

- We assume that the total plasma current may change, but the distribution does not, hence $\mathbf{I_x}(t) = (\mathbf{I_{x0}}/I_{p0})I_p(t)$
- We write the mutual inductance between plasma and other conductors as:

$$M_{pa_k} = \frac{\sum_x M(r_x, z_x, a_k) j(r_x, z_x) S}{\sum_x j(r_x, z_x) S} = \frac{\mathbf{I_x^T M_{xa}}}{I_p} \quad (1)$$

  Here $M(r_x, z_x, a_k)$ is the mutual inductance between active coil a_i and the filament on the grid at point (r_x, z_x) and $S = \Delta r \Delta z$ is the surface of the current-carrying element
- Same for M_{pv_k} between plasma and vessel filaments.
- Self-inductance of the plasma: for a single conductor $W_{mag} = \frac{1}{2} L I^2$, while for multiple conductors $W_{mag} = \frac{1}{2} \mathbf{I}^T \mathbf{M} \mathbf{I}$, therefore $L_p = \mathbf{I_x}^T \mathbf{M_{xx}} \mathbf{I_x} / I_p^2$

Circuit equations including rigid plasma

- The circuit equation for the plasma is:

$$0 = \mathbf{M_{pa}}\dot{\mathbf{I}}_{\mathbf{a}} + \mathbf{M_{pu}}\dot{\mathbf{I}}_{\mathbf{u}} + L_p\dot{I}_p + R_p I_p \tag{2}$$

  Where
  - L_p is the plasma self-inductance
  - R_p is the plasma electrical resistance.
  - $\mathbf{M_{pa}} = [M_{pa_1}, \dots, M_{pa_{n_a}}]$ is the row vector containing mutuals between plasma and active coils.
  - $\mathbf{M_{pu}} = [M_{pu_1}, \dots, M_{pu_{n_u}}]$ is the row vector of mutuals between plasma and passive currents.
- We can now include the plasma in the active coil and passive circuit equation

$$\begin{pmatrix} \mathbf{V_a} \\ \mathbf{0} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{M_{aa}} & \mathbf{M_{au}} & \mathbf{M_{ap}} \\ \mathbf{M_{ua}} & \mathbf{M_{uu}} & \mathbf{M_{up}} \\ \mathbf{M_{pa}} & \mathbf{M_{pu}} & L_p \end{pmatrix} \begin{pmatrix} \dot{\mathbf{I}}_{\mathbf{a}} \\ \dot{\mathbf{I}}_{\mathbf{u}} \\ \dot{I}_p \end{pmatrix} + \begin{pmatrix} \mathbf{R_{aa}} & 0 & 0 \\ 0 & \mathbf{R_{uu}} & 0 \\ 0 & 0 & R_p \end{pmatrix} \begin{pmatrix} \mathbf{I_a} \\ \mathbf{I_u} \\ I_p \end{pmatrix} \tag{3}$$

Plasma current induction by ohmic transformer

- We see directly from the plasma circuit equation

$$0 = \mathbf{M_{pa}}\dot{\mathbf{I}}_\mathbf{a} + \mathbf{M_{pu}}\dot{\mathbf{I}}_\mathbf{u} + L_p\dot{I}_p + R_pI_p \tag{4}$$

  that we must *induce* a voltage to drive the plasma current via PF coils: $\mathbf{M_{pa}}\dot{\mathbf{I}}_\mathbf{a}$.
- In practice, a combination of PF coils called the 'ohmic' coils is used to drive I_p. This set is sometimes also called the Central Solenoid (CS).
- Assume we want to drive a constant current I_p, with a collection of Ohmic coils carrying current I_{oh}. Then we need to satisfy

$$0 = M_{p,oh}\dot{I}_{oh} + R_pI_p \tag{5}$$

  Hence $-M_{p,oh}\dot{I}_{oh} = R_pI_p$. This quantity is sometimes called the 'loop voltage' used to drive the plasma current inductively.

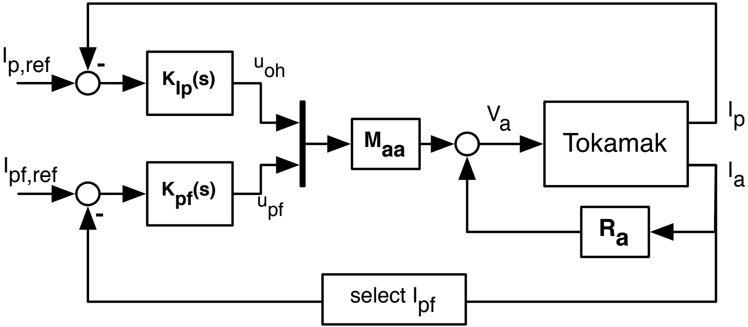
Limits of operation for inductively driven plasmas

- **The OH current must continuously change to drive the plasma current.**
- $I_{oh,min} \rightarrow I_{oh,max}$ or vice versa.
- This creates *flux swing* in the plasma of $\Delta\Psi_{oh} = \pm M_{p,oh}(I_{oh,max} - I_{oh,min})$. This is a measure of how long a tokamak PF coil set can sustain an Ohmic plasma.
- A cold plasma is more resistive than a hot plasma. So a hot plasma can be sustained longer for a given $\Delta\Psi_{oh}$.
- Tokamaks are inherently *pulsed* devices.
- For steady-state fusion reactor we need another, *non-inductive* means to drive the current (later in the course).

Controller design for plasma current

- Recall the coil current controller of the form $\mathbf{V_a} = \mathbf{M_{aa}}\mathbf{u} + \mathbf{R_a}\mathbf{I_a}$ which allows us to write $\mathbf{I_a} = \int_0^t \mathbf{u}(t)\mathrm{d}t$.
- We now choose the 'OH' coils to be used for I_p control.

$$\mathbf{u} = \begin{pmatrix} u_{oh} \\ u_{pf} \end{pmatrix} = \begin{pmatrix} -K_{I_p}(s)(I_{p,ref} - I_p) \\ \mathbf{K_{pf}}(s)(\mathbf{I_{pf,ref}} - \mathbf{I_{pf}}) \end{pmatrix} \tag{6}$$



Controller design for plasma current

- This gives $\dot{I}_{oh} = -K_{I_p}(s)(I_{p,ref} - I_p)$
- The closed-loop transfer function from $I_{p,ref}$ to I_p can be calculated, neglecting vessel currents:

$$0 = L_p \dot{I}_p + M_{p,oh} \dot{I}_{oh} + R_p I_p \tag{7}$$

$$M_{p,oh} K_{I_p}(s)(I_{p,ref} - I_p) = L_p s I_p + R_p I_p \tag{8}$$

$$M_{p,oh} K_{I_p}(s)(I_{p,ref}) = \left(L_p s + R_p + M_{p,oh} K_{I_p}(s)\right) I_p \tag{9}$$

$$\frac{I_p(s)}{I_{p,ref}(s)} = \frac{\frac{M_{p,oh}}{L_p} K_{I_p}(s)}{\left(s + \frac{R_p}{L_p} + \frac{M_{p,oh}}{L_p} K_{I_p}(s)\right)} \tag{10}$$

Controller design for plasma current - 2

- Closed-loop transfer function:

$$\frac{I_p(s)}{I_{p,ref}(s)} = \frac{\frac{M_{p,oh}}{L_p} K_{I_p}(s)}{(s + \frac{R_p}{L_p} + \frac{M_{p,oh}}{L_p} K_{I_p}(s))} \tag{11}$$

- If we choose a proportional controller $K_{I_p}(s) = k_p$ and compute the steady-state gain

$$\lim_{t\to\infty} I_p(t) = \lim_{s\downarrow 0} \frac{I_p(s)}{I_{p,ref}(s)} = \frac{M_{p,oh}k_p}{(R_p + M_{p,oh}k_p)} \tag{12}$$

- For high gain and/or low-resistivity plasmas, this approaches 1

Section 2

Magnetic measurements

Magnetic measurements

Various methods are used to measure currents and fields in around the tokamak. These are also used to reconstruct the plasma position and shape (later).
References: [1], [2], [3], [4]

Magnetic measurements - Magnetic probes

- A magnetic probe constists of a small winding with several turns.
- Assuming the probe is small so that the field is homogeneous inside it, the voltage measured by the probe is

$$V_{probe} = NA\dot{B} \tag{13}$$

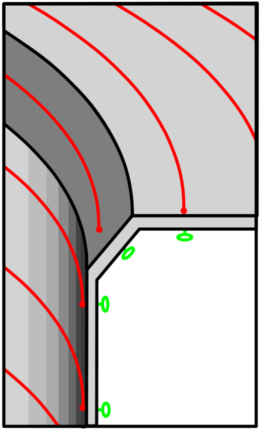
where N is the number of turns, A is the area of the probe and $\dot{B}$ is the time derivative of the field. To get the total field B the raw voltage signal must be integrated.

- Advantages: Simple, cheap, robust.
- Disadvantages: Integrator drift, no absolute measurement, damage and disturbance from fusion neutrons.

Magnetic measurements - Flux loops

- Flux loops consist of a wire wound around the torus. This measures
$$U_f = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \iint \frac{\mathrm{d}B}{\mathrm{d}t} dS_f \tag{14}$$
where S_f is the surface enclosed by the loop.
- The signal is integrated in time to yield the flux enclosed by the loop
$$\psi_f = \int_t U_f(t')dt' \tag{15}$$
- Ideally, a flux loop wound around the torus at coordinates (r_f, z_f) measures the local poloidal flux at that point.
- In practice, flux loops are often not ideal and need to be guided around ports etc, so need to compensate for these 3D effects.

Magnetic measurements - Illustration



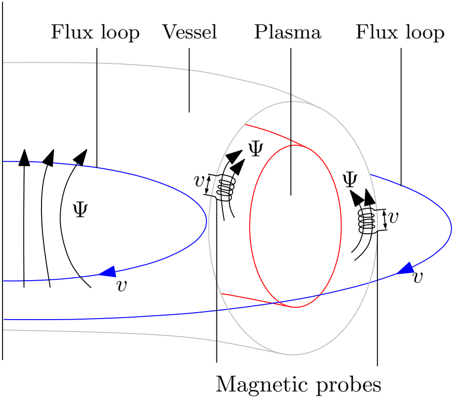


Figure: From C. Gootzen MSc thesis, TU/e 2014

Magnetic measurements - Illustration

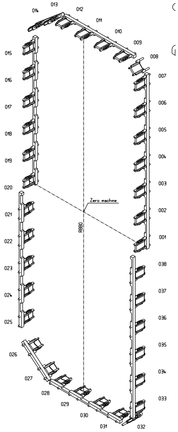
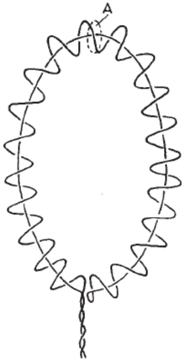


Figure: TCV poloidal magnetic probe array

Magnetic measurements - Rogowski coil

- A Rogowski coil measures the spatial integral of field through solenoid that is 'closed' on itself: $V_R = n \oint \left(\int dA \right) \dot{\mathbf{B}} \cdot \mathbf{d}\ell$ where n is the number of windings per unit length, A is the cross-sectional area of the solenoid, and $\mathbf{d}\ell$ an infinitesimal element along the solenoid axis.
- Recalling Ampère's law: $\oint \mathbf{B} \cdot \mathbf{d}\ell = \mu I$ we see that this measures $V_R = nA \frac{\mathrm{d}}{\mathrm{d}t} I$.
- The voltage is (electronically) integrated to get the current itself: $I = \frac{1}{nA} \int V_R dt$.
- Rogowski coils are frequently used to measure the total plasma current.

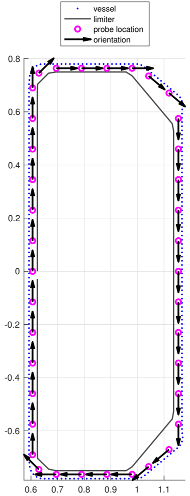


Magnetic measurements - plasma current estimators

- For practical reasons, no Rogowski coil was installed in the TCV tokamak
- An alternative plasma current estimator can be constructed by numerically integrating the (tangential) magnetic probe measurements, using the trapeze rule:

$$\mu_0 I_p = \oint B_p d\ell \approx \sum_{i=1}^{n_m} \frac{1}{2}(B_m^{i-1} + B_m^i)\Delta s_i \quad (16)$$

with $\Delta s_i = \sqrt{(r_i - r_{i-1})^2 + (z_i - z_{i-1})^2}$, and closing the contour with $\text{probe}_0 = \text{probe}_{n_m}$



Magnetic measurements - Diagmagnetic loop

- The plasma also carries poloidal currents which generate a toroidal field and flux.
- The total plasma toroidal flux ϕ_p is related to the plasma pressure via the approximate relation

$$\phi_p = \frac{\mu_0^2 I_p^2}{8\pi B_t}(1-\beta_p) \tag{17}$$

  with B_t the toroidal field and β_p the polodidal-field normalized pressure (later)
- Plasma toroidal flux in TCV: $0.04\mathrm{mWb}$ to be separated from toroidal field flux $2\mathrm{Wb}$ and induced vessel poloidal currents by complex calibration and signal processing [4].

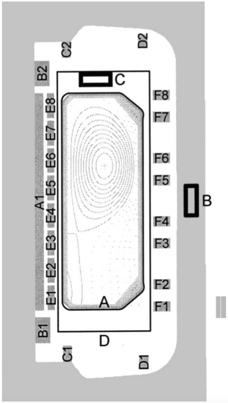
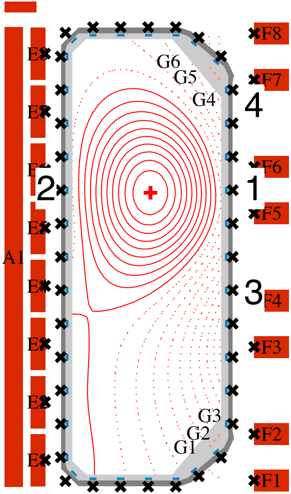


Figure: Four loops in the poloidal plane used for TCV DML diagnostic, from [4]

Estimating R,Z from measurements: the crude way

- Choose B probes and flux loops close to the plasma at appropriate locations
- Difference between fluxes/fields at each location are rough estimate of plasma position. E.g.:

$$\delta R I_p \approx k_r(\psi_1 - \psi_2) \quad (18)$$
$$\delta Z I_p \approx k_z(B_{p3} - B_{p4}) \quad (19)$$



Estimating R,Z from measurements

- A slightly less crude way is to combine magnetic probes and flux loops to extrapolate the flux.
- Assume in nominal case $\psi_o = \psi_i$
- $\tilde{\psi}_o - \tilde{\psi}_i = \left(\frac{\partial \psi}{\partial r}\Big|_o^{eq} - \frac{\partial \psi}{\partial r}\Big|_i^{eq} \right) \Delta r$
- Estimate for flux at position i, o:
  $\tilde{\psi}_i \approx \psi_A + \frac{\partial \psi}{\partial r}\Big|_A (r_i - r_A)$,
  $\tilde{\psi}_o \approx \psi_B - \frac{\partial \psi}{\partial r}\Big|_B (r_B - r_o)$, with
  $\frac{\partial \psi}{\partial r}\Big|_A = 2\pi r_A B_{z,A}$ $\frac{\partial \psi}{\partial r_B}\Big|_B = 2\pi r B_{z,B}$
- Solve for ΔR with measurements $B_{z,A}, B_{z,B}, \psi_A, \psi_B$

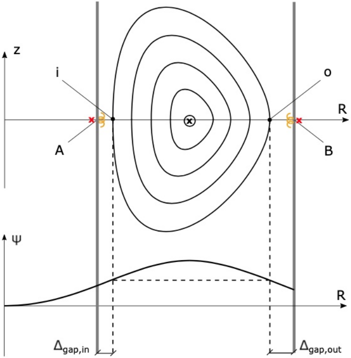


Figure: From F. Pesamosca EPFL PhD thesis 2021

Measurement equation

- Since all these diagnostics measure fluxes or fields, which are all linear functions of currents in the coils, vessel and plasma...
- We can write the measurement output equations of the

$$\mathbf{B^{probe}} = \mathbf{B_{ma}I_a} + \mathbf{B_{mv}I_v} + \mathbf{B_{mx}I_x} \tag{20}$$

$$\psi^{\mathbf{loop}} = \mathbf{M_{fa}I_a} + \mathbf{M_{fv}I_v} + \mathbf{M_{fx}I_x} \tag{21}$$

$$\mathbf{I^{meas}} = \mathbf{S_{Ia}I_a} + \mathbf{S_{Iv}I_v} \tag{22}$$

- Defining $\mathbf{x} = \begin{bmatrix} \mathbf{I_a} & \mathbf{I_v} & \mathbf{I_x} \end{bmatrix}$ and collecting the measurement vector in $\mathbf{y}$ we can write this as a matrix equation of the form

$$\mathbf{y} = \mathbf{Cx} \tag{23}$$

Can we attempt to solve this weighted least-squares problem?

$$\min_x \|\mathbf{W}(\mathbf{y} - \mathbf{Cx})\| \tag{24}$$

Estimating current distribution from measurements

- The coil currents $\mathbf{I_a}$ are usually measured directly
- The passive (e.g. vessel) currents $\mathbf{I_v}$ are more difficult to measure. Two approaches:
  - Reduce the number of free parameters by choosing an eigenvector or other parametrization and attempt to estimate it.
  - Associate a flux loop with a segment of the vessel and estimate its current $I_s = U_f/R_s$ with U_f voltage measurement of the flux loop measurement.
- The current distribution inside the plasma can not be measured directly. Estimating $\mathbf{I_x}$ is not possible since there are many more elements than there are measurements. **The least-squares problem is ill-conditioned**. Need, either:
  - Regularization (impose structure on the solution, e.g. smoothness)
  - Reduction of the number of free parameters: choose coarser grid for $\mathbf{I_x}$, parametrize the current distribution, or impose that the solution must represent an MHD equilibrium (later in the course).

Simple estimators for the current distribution

- Choose a parametrization of the current plasma distribution, for example
  - Choose discrete filaments carrying currents $I_x = T_{xh} I_h$ with T_{xh}
  - Choose finite elements distributed on the plasma grid: $I_x = T_{xh} I_h$ with T_{xh} representing a spatially distributed current.
- We can then express the mag. probe and flux loop measurements, neglecting the vessel curents, as

$$\begin{bmatrix} \hat{B}_m \\ \hat{\psi}_f \end{bmatrix} = \begin{bmatrix} B_{ma} \\ M_{fa} \end{bmatrix} I_a + \begin{bmatrix} B_{mh} \\ M_{fh} \end{bmatrix} I_h$$

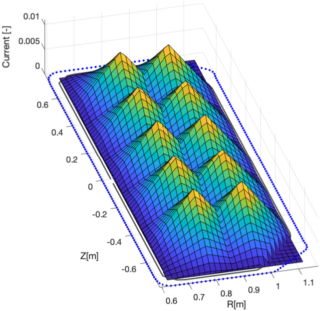


Figure: Illustration of 10 Bilinear Finite Elements for TCV

Simple estimators for the current distribution

- Typically $\sim$100 measurements are available, so 100 equations
- Typically 4-8 degrees of freedom for the plasma can be used
- Solve in the least-squares sense: minimise w.r.t. I_h the function

$$\chi^2 = \sum_{i=1}^{n_m} \frac{(B_m^i - \hat{B}_m^i(I_h))^2}{e_{B_m}^2} + \sum_{i=1}^{n_f} \frac{(\psi_f^i - \hat{\psi}_f^i(I_h))^2}{e_{\psi_f}^2} \tag{25}$$

  where e_{B_m} , e_{ψ_f} are the expected measurement errors, used to weigh the contributions to the χ^2 term.
- Exercise: write (25) in the form $\chi^2 = \|Ax - b\|_2^2$

Simple estimators for the current distribution

- Typically $\sim$100 measurements are available, so 100 equations
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  where e_{B_m} , e_{ψ_f} are the expected measurement errors, used to weigh the contributions to the χ^2 term.
- Exercise: write (25) in the form $\chi^2 = \|Ax - b\|_2^2$
- Solution:
  - $x = I_h$, $A = W \begin{bmatrix} B_{mh} \\ M_{fh} \end{bmatrix}$, $b = W \begin{bmatrix} B_m - B_{ma} I_a \\ \psi_f - M_{fa} I_a \end{bmatrix}$, with

    $W = \begin{bmatrix} 1/e_{B_m} & 0 \\ 0 & 1/e_{\psi_f} \end{bmatrix}$

Simple estimators for the current distribution

- Exercise: solve the least-squares problem

$$\min_x \|Ax - b\|_2^2 \tag{26}$$

Simple estimators for the current distribution

- Exercise: solve the least-squares problem

$$\min_x \|Ax - b\|_2^2 \tag{26}$$

- Solution:
  - We minimize $J(x) = \chi^2 = (Ax - b)^T(Ax - b) = x^TA^TAx - 2x^TA^Tb + b^Tb$ with respect to x. Write the Jacobian:

$$\frac{\partial J}{\partial x} = 2A^TAx - 2A^Tb \tag{27}$$

    The function has a minimum where $\frac{\partial J}{\partial x} = 0$. This yields:

$$x = A^+b \tag{28}$$

    where $A^+ = \left(A^TA\right)^{-1}A^T$ is the Moore-Penrose Pseudoinverse of A
  - in matlab: `x=A\b` or `x=pinv(A)*b`

Simple estimators for the current distribution

- We obtain

$$I_h = \underbrace{\left(W \begin{bmatrix} B_{mh} \\ M_{fh} \end{bmatrix} \right)^{+} W}_{=Q} \begin{bmatrix} B_m - B_{ma} I_a \\ \psi_f - M_{fa} I_a \end{bmatrix} \quad (29)$$

- This solution can be cast into a linear estimator:

$$I_h = \underbrace{\begin{bmatrix} A_{hm} & A_{hf} & A_{ha} \end{bmatrix}}_{\text{pre-computed}} \underbrace{\begin{bmatrix} B_m \\ \psi_f \\ I_a \end{bmatrix}}_{\text{measured}} \quad (30)$$

with $\begin{bmatrix} A_{hm} & A_{hf} \end{bmatrix} = Q$ and $A_{ha} = -Q \begin{bmatrix} B_{ma} \\ M_{fa} \end{bmatrix}$

Current distribution moments

- Total plasma current:

$$I_p = \sum_x I_x = \sum_x T_{xh} I_h = \left[\begin{array}{ccc} \sum_x T_{xh} A_{hm} & \sum_x T_{xh} A_{hf} & \sum_x T_{xh} A_{ha} \end{array} \right] \left[\begin{array}{c} B_m \\ \psi_f \\ I_a \end{array} \right] \tag{31}$$

- Actually not the best estimator, the numerical estimator (16) is more accurate, particularly if we also subtract the contribution from poloidal field coils:

$$I_p = \sum_{i=1}^{n_m} \frac{1}{2} (B_m^{i-1} + B_m^i - (B_{ma}^{i-1} I_a + B_{ma}^i I_a)) \Delta s_i \tag{32}$$

Current distribution moments

- Vertical position (current distribution centroid):

$$zI_p = \sum_x z_x I_x = \left[\begin{array}{ccc} \sum_x z_x T_{xh} A_{hm} & \sum_x z_x T_{xh} A_{hf} & \sum_x z_x T_{xh} A_{ha} \end{array} \right] \left[\begin{array}{c} B_m \\ \psi_f \\ I_a \end{array} \right]$$

- or: error w.r.t. a given reference z_0

$$\begin{aligned} &\sum_x (z_x - z_0) I_x \\ &= \left[\begin{array}{ccc} \sum_x (z_x - z_0) T_{xh} A_{hm} & \sum_x (z_x - z_0) T_{xh} A_{hf} & \sum_x (z_x - z_0) T_{xh} A_{ha} \end{array} \right] \left[\begin{array}{c} B_m \\ \psi_f \\ I_a \end{array} \right] \end{aligned} \tag{33}$$

- Radial position (current distribution centroid):

$$rI_p = \sum_x r_x I_x = \left[\begin{array}{ccc} \sum_x r_x T_{xh} A_{hm} & \sum_x r_x T_{xh} A_{hf} & \sum_x r_x T_{xh} A_{ha} \end{array} \right] \left[\begin{array}{c} B_m \\ \psi_f \\ I_a \end{array} \right]$$

Current distribution moments

- Elongation estimator:

$$\sum_x (z_x - z_0)^2 I_x =$$

$$\left[\begin{array}{ccc} \sum_x (z_x - z_0)^2 T_{xh} A_{hm} & \sum_x (z_x - z_0)^2 T_{xh} A_{hf} & \sum_x (z_x - z_0)^2 T_{xh} A_{ha} \end{array} \right] \left[\begin{array}{c} B_m \\ \psi_f \\ I_a \end{array} \right]$$

Generalized toroidal current distribution estimator

- Write a more general least-squares estimator to include passive current contributions I_u and allow uncertainties in I_a measurements
- Recall we define segment currents as $I_s = S_{sv} I_v$ (where S_{sv} is a selection matrix), and estimate these currents as $I_s^{estim} = U_f / R_s$.

$$\begin{bmatrix} \hat{B}_m \\ \hat{\psi}_f \\ \hat{I}_a^{meas} \\ \hat{I}_s^{estim} \end{bmatrix} = \begin{bmatrix} B_{ma} \\ M_{fa} \\ \mathbb{I}_a \\ 0 \end{bmatrix} I_a + \begin{bmatrix} B_{mu} \\ M_{fu} \\ 0 \\ S_{sv} T_{vu} \end{bmatrix} I_u + \begin{bmatrix} B_{mh} \\ M_{fh} \\ 0 \\ 0 \end{bmatrix} I_h$$

This can again be solved in a weighted least-squares sense, with uncertainties ΔB_m, $\Delta \psi_f$, ΔI_s, ΔI_a, yielding

$$\begin{bmatrix} I_h \\ I_a \\ I_u \end{bmatrix} = \begin{bmatrix} A_{hm} & A_{hf} & A_{ha} & A_{hU} \\ A_{am} & A_{af} & A_{aa} & A_{aU} \\ A_{sm} & A_{sf} & A_{sa} & A_{sU} \end{bmatrix} \begin{bmatrix} B_m \\ \psi_f \\ \hat{I}_a^{meas} \\ U_f \end{bmatrix}$$

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