Introduction to holography - Lecture IX

Last time: radial quantiz, state-op. corresp, unitarity bounds

- OPE, how n-pct. f. can be reduced to lower-pct. f, conformal bootstrap

This time: brief discussion of Weyl vs. conformal vs. scale invar

- trace / conformal anomaly
- applications in $d=2$: corr. f. of stress tensor, the Schwarzian derivative, Casimir eng. on the cylinder, Cardy's formula

Weyl vs conformal vs scale invariance

(i.e. global scale just an $\int T^\mu{}_\mu = 0$)

- **Weyl**: partition function $\underbrace{Z[g]}_{\int \mathcal{D}\phi \, e^{-S[\phi,g]}}$ (assumed to be diff. invar) invariant under $g_{\mu\nu}(x) \to \Omega^2(x)\, g_{\mu\nu}(x)$, $\forall \underbrace{\Omega(x)}_{\text{local}}$

  - defining the stress tensor as $T_{\mu\nu} = \dfrac{2}{\sqrt{g}} \dfrac{\delta S_E}{\delta g^{\mu\nu}}$ (euclidean)

  - then $\delta \ln Z = -\frac{1}{2}\int d^dx \sqrt{g} \langle T_{\mu\nu}\rangle_g \, \delta g^{\mu\nu} = \int d^dx \sqrt{g} \, \langle T^\mu{}_\mu\rangle_g \, \sigma(x)$

    $\Omega = e^\sigma \simeq 1 + \sigma(x)$

  - invariance $\Rightarrow$ $T^\mu{}_\mu = 0$. (up to contact terms).

  - correlation functions of primary ops (def^n) transform as

$$\langle O_1(x_1) \dots O_n(x_n)\rangle_{\underbrace{\Omega^2 g}_{\text{corr. f in} \neq \text{backgnd. metrics}}} = \Omega(x_1)^{-\Delta_1} \dots \Omega(x_n)^{-\Delta_n} \langle O_1(x_1) \dots O_n(x_n)\rangle_g$$

  - under an $\frac{1}{\infty}$ change in the metric $\langle O_1(x_1)\dots O_n(x_n)\rangle_{g+\delta g} - \langle O_1(x_1) \dots O_n(x_n)\rangle_g$

$$= -\frac{1}{2}\int d^dx \sqrt{g}\, \delta g^{\mu\nu} \langle T_{\mu\nu}\, O_1(x_1) \dots O_n(x_n)\rangle_g$$

- taking $\delta g_{\mu\nu} = 2\sigma(x)\, g_{\mu\nu}$ & noting that $\delta_\sigma \langle O_1 \dots O_n \rangle = -\sum_i \Delta_i \sigma(x_i) \langle O_1 \dots O_n \rangle$

we derive the Ward identity

$$\langle T^\mu{}_\mu(x)\, O_1(x_1) \dots O_n(x_n) \rangle = -\sum_{i=1}^n \Delta_i \delta^{(d)}(x-x_i) \langle O_1(x_1) \dots O_n(x_n) \rangle$$

- notice this may be further used to derive the conformal Ward id, using the fact that conf transf = coord transf + (special) Weyl transf:

under a coord transf $x^\mu \to \tilde{x}^\mu = x^\mu + \varepsilon^\mu$ & $g_{\mu\nu} \to \tilde{g}_{\mu\nu} - \nabla_\mu \varepsilon_\nu - \nabla_\nu \varepsilon_\mu$

the correlation f. are invar (for scalars)

$$\langle O_1(\tilde{x}_1) \dots O_n(\tilde{x}_n) \rangle_{\tilde{g}} = \langle O_1(x_1) \dots O_n(x_n) \rangle_g$$

$$\langle O_1(x_1) \dots O_n \rangle_g + \sum_i \varepsilon^\mu(x_i) \partial_{\mu i} \langle O_1 \dots O(x_i) \dots O_n(x_n) \rangle + \langle O_1 \dots O_n \rangle_g - \frac{1}{2} \int d^d x \sqrt{g}\, \underbrace{\delta g^{\mu\nu}}_{2\nabla^\mu \varepsilon^\nu} \langle T_{\mu\nu} O_1 \dots O_n \rangle_g$$

arbitrary ε

+ fast decay $\Rightarrow$ $$\sum_i \varepsilon^\mu(x_i) \frac{\partial}{\partial x_i^\mu} \langle O_1(x_1) \dots O_i(x_i) \dots O_n(x_n) \rangle_g = -\int d^d x \sqrt{g}\, \varepsilon^\nu \langle \nabla_\mu T^\mu{}_\nu(x) O_1 \dots O_n \rangle_g$$

- consider now the $\int$ over a surface ∂B = bnd of region B.

$$\int_{\partial B} d^{d-1}x \sqrt{\sigma}\, n_\mu \varepsilon_\nu(x) \langle T^{\mu\nu}(x) O_1 \dots O_n \rangle = \int_B d^d x \sqrt{g}\, \nabla_\mu \left(\varepsilon_\nu \langle T^{\mu\nu} O_1 \dots O_n \rangle \right) =$$

$$= \int_B d^d x \sqrt{g} \left\{ \underbrace{\nabla_\mu \varepsilon_\nu(x)}_{\frac{1}{d} \nabla_\lambda \varepsilon^\lambda g_{\mu\nu} \text{ for conformal}} \langle T^{\mu\nu}(x) O_1 \dots O_n \rangle + \varepsilon_\nu(x) \langle \nabla_\mu T^{\mu\nu}(x) O_1 \dots O_n \rangle \right\}$$

then use Weyl ward id

$$= \int_B d^d x \sqrt{g} \left\{ \frac{1}{d} \nabla_\lambda \varepsilon^\lambda \underbrace{\langle T^\mu{}_\mu(x) O_1 \dots O_n \rangle}_{-\sum_i \Delta_i \delta(x-x_i) \langle O_1 \dots O_n \rangle} - \varepsilon^\mu(x) \frac{\partial}{\partial x^\mu} \langle O_1 \dots O_n \rangle \right\}$$

$$= -\sum_{i=1}^n \left(\varepsilon^\mu(x_i) \frac{\partial}{\partial x_i^\mu} + \Delta_i \frac{1}{d} \nabla_\lambda \varepsilon^\lambda(x_i) \right) \langle O_1(x_1) \dots O_n(x_n) \rangle_g$$

$$= \sum_i \langle O_1(x_1) \dots [Q_\varepsilon, O_i(x_i)] \dots O_n(x_n) \rangle$$

same as we used before

- to sum up, Weyl invar. requires $T^\mu{}_\mu = 0$, up to contact terms & up to anomalies

- also note Weyl (+ diff invar, which we assume) $\Rightarrow$ conformal invar

In a conformally invar theory, the conformal generators must take the form

local ops w/no explicit coord. dep. — symm — w.l.o.g

$$J^\mu = T^\mu{}_\nu \, \varepsilon^\nu(x) + K^\mu \frac{\nabla_\lambda \varepsilon^\lambda}{d} + \nabla_\nu\Big(\underbrace{\frac{\nabla_\lambda \varepsilon^\lambda}{d}}_{\text{conf. transf. vectors}}\Big) L^{\mu\nu}$$

- the argument is as follows: the charges G assoc w/ these symm are cons.

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + i[\underbrace{H}_{P^0}, G] = 0$$

- for $G \neq P^\mu$, this eqn. det. the coordinate-dependent part of the generator

(e.g., for $G = D$, we find $\partial_t D = \pm i H = \pm i \int T_{00} \, d^{d-1}x$

correct?

+ Lorentz invar & locality $\Rightarrow$ 1^{st} term. + K^μ etc.

- J^μ conservation $\Rightarrow$ $$T^{\mu\nu} \underbrace{\nabla_\mu \varepsilon_\nu}_{\frac{1}{d}\nabla_\lambda \varepsilon^\lambda g_{\mu\nu}} + \nabla_\mu K^\mu \frac{\nabla_\lambda \varepsilon^\lambda}{d} + K^\mu \nabla_\mu\Big(\frac{\nabla_\lambda \varepsilon^\lambda}{d}\Big) + \nabla_\nu\Big(\frac{\nabla_\lambda \varepsilon^\lambda}{d}\Big) \nabla_\mu L^{\mu\nu} + \underbrace{\nabla_\mu \nabla_\nu \Big(\frac{\nabla_\lambda \varepsilon^\lambda}{d}\Big)}_{0 \text{ for conformal global}} L^{\mu\nu} = 0.$$

$$\Rightarrow \begin{cases} T^\mu{}_\mu = -\partial_\mu K^\mu \\ K^\mu = -\partial_\nu L^{\mu\nu} \end{cases}$$

&, in $d=2$, the last term $\neq 0$ unless $L^{\mu\nu} \propto g^{\mu\nu}$ since need $\Box \, \partial \varepsilon$

- therefore, the condition for conformal invar is $T^\mu{}_\mu = \partial_\mu \partial_\nu L^{\mu\nu}$ $(d>2)$

$(= \Box L \quad (d=2))$

- by contrast, the condition for scale invar is just

$T^\mu{}_\mu = -\partial_\mu K^\mu$ strictly weaker; however, often no candidate

dim $d-1$ $\nearrow$ K_μ w/ req. prop. (need unitarity)

- given this form of the trace, then one can define an improved stress t.

$$\Theta_{\mu\nu} = T_{\mu\nu} + \frac{1}{d-2}\left(\partial_\mu\partial_\alpha L^\alpha{}_\nu + \partial_\nu\partial_\alpha L^\alpha{}_\mu - \Box L_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta L^{\alpha\beta}\right)$$

$$+ \frac{1}{(d-1)(d-2)}\left(\eta_{\mu\nu}\Box L^\alpha{}_\alpha - \partial_\mu\partial_\nu L^\alpha{}_\alpha\right) \qquad d>2$$

$$\Theta_{\mu\nu} = T_{\mu\nu} - \left(\eta_{\mu\nu}\Box L - \partial_\mu\partial_\nu L\right) \qquad d=2$$

- similar considerations for conformal $\Rightarrow$ Weyl.

- note all these improvement terms (& the improvement ambiguity of the form $\Delta T_{\mu\nu} \sim \partial^\rho\partial^\sigma Y_{\mu\rho\nu\sigma}$) can be generated via the curvature couplings
$$S_{imp} \sim \int d^dx \sqrt{g}\,\left(RL + R_{\mu\nu}L^{\mu\nu} + R_{\mu\nu\rho\sigma}Y^{\mu\nu\rho\sigma}\right).$$

- in 2d, note that $T^\mu{}_\mu = 2T_{z\bar z} + 2T_{\bar z z} = 0$
$\Rightarrow$ T_{zz} is holomorphic & $T_{\bar z\bar z}$ is anti-holom

Conformal / Weyl anomalies

- Weyl-invar theories are automatically conformally invar when restricted to flat sp.

- this is guaranteed by the vanishing of the trace of the stress tensor (in flat space)

- however, one may have a situation in which $T^\mu{}_\mu$ is a function of the backgnd. fields (i.e., the metric), times the identity operator $\Rightarrow$ anomalous breaking of Weyl symm. (cls symm. that is broken by quantum effects)

- since $T^\mu{}_\mu(x)$ is a local, scalar operator of dimension d, so should be the RHS (& must vanish in flat sp) & be diff. invar & consistent w/ abelian nature of Weyl transf)

  - $d=2$ $\qquad T^\mu{}_\mu = +\frac{c}{24\pi} R[g]$ (eucl. sign, $-$ in Lorentzian) — central charge

  - $d=4$ $\qquad T^\mu{}_\mu = \frac{a}{64\pi^2}E_4 + \frac{c}{64\pi^2}C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} + e_1 R^2 + e_2 \Box R.$

where $E_4 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Euler density (topological)

and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ is the square of the Weyl tensor

- the Wess-Zumino consistency cond $[\delta_{\sigma_1}, \delta_{\sigma_2}] \ln Z = \int d^4x \sqrt{g} \left(\sigma_2(x)\delta_{\sigma_1} \underbrace{A(x)}_{\text{anomalous } T^\mu_\mu} - \sigma_1 \delta_{\sigma_2} A\right) = 0$

sets the coeff $e_1 = 0$, & $\Box R$ is trivial (shift of stress t.).

d=2

- in $d=2$, $\forall$ metric is rel to $\delta_{\mu\nu}$ by a diffeo + conformal transf. Fixing diffeos

we can write

$$g_{\mu\nu}(x) = e^{2\sigma(x)}\delta_{\mu\nu} \qquad w/ \quad R = -2e^{-2\sigma}\Box_\delta \sigma(x)$$

- since the Weyl anomaly $\Rightarrow$ dep. of path int. on σ $\Rightarrow$ completely det. dependence of Z on the 2d metric.

$$\delta \ln Z[e^{2\sigma}\delta_{\mu\nu}] = \int d^2x\, e^{2\sigma} \underbrace{\langle T^\mu{}_\mu\rangle}_{+\frac{c}{24\pi}R} \delta\sigma = -\frac{c}{12\pi}\int d^2x\, \Box_\delta\sigma\, \delta\sigma$$

$$\Rightarrow \quad Z[e^{2\sigma}\delta_{\mu\nu}] = e^{-\frac{c}{24\pi}\int d^2x\, \sigma(x)\Box_\delta\sigma(x)} Z[\delta_{\mu\nu}] .$$

how the partition f. dep. on the scale factor

- the prefactor can be rewritten in covariant form by noting that $R = -2\Box_g\sigma(x)$

$$\Rightarrow \quad \sigma(x) = \frac{1}{2}\int d^2x' \sqrt{g(x')}\, G(x,x')\, R(x') \qquad \Box_g G(x,x') = -\frac{1}{\sqrt{g}}\delta^{(2)}(x-x')$$

$$\Rightarrow \quad Z[g_{\mu\nu}] = Z[\delta_{\mu\nu}]\, e^{\frac{c}{96\pi}\int d^2x\sqrt{g}\int d^2x' \sqrt{g(x')}\, R(x)\, G(x,x')\, R(x')}$$

Polyakov eff. action.

non-local $\sim \frac{1}{\Box}$

$\Rightarrow$ all corr. f. of the stress t. are completely fixed by c

- e.g. for T_{zz}, which couples to $h_{\bar z\bar z}$, the relevant term is $R \sim \partial^2 h_{\bar z \bar z}$ $\quad G \sim \ln|z-\bar z|^2$

$\langle T(z) T(0)\rangle = \frac{c/2}{z^4}$

· another thing one may deduce is how $T_{\mu\nu}$ transforms under conformal transf.

· in the exercise you are asked to show that $\langle T_{zz} \rangle_{e^{2\sigma}\delta_{\mu\nu}} = \frac{c}{12\pi}\left[\partial_z^2\sigma - (\partial_z\sigma)^2\right]$

do finite directly $\quad e^{2\sigma} = \partial_z \tilde{z}$

• remember in 2d conf. transf. are given by $z \to z + f(z) \quad \bar{z} \to \bar{z} + \bar{f}(\bar{z})$ (infinitesimal)

under such a transf. $\delta g_{z\bar{z}} = -\partial_z \xi_{\bar{z}} - \partial_{\bar{z}} \xi_z = -\frac{1}{2}(f' + \bar{f}') = -2\delta\sigma \cdot \frac{1}{2}$

$z, \bar{z}$: complex coord on $\mathbb{R}^2$ / lightlike coord. in $\mathbb{R}^{1,1}$

note, in part, that $\partial\bar{\partial}\delta\sigma = 0$, so no trace is induced for $T^\mu{}_\mu$.

· from the above $\delta_f T_{zz} = \underbrace{\frac{c}{24\pi} f'''}_{\text{anomalous transf. law.}} - 2f' T_{zz} - \underbrace{f\partial_z T_{zz}}_{\text{effect of translation}}$ normalization! (-2π)

· in fact, in 2d it is useful to distinguish general conformal transf, param. by arbitrary $f(z)$, $\bar{f}(\bar{z})$ (infinitesimal) from global conformal transf, which exponentiate to well-defined generators on the entire S^2 (Möbius transf)

$$\underset{\text{finite!}}{f(z)} = \frac{az+b}{cz+d} \quad \text{w/ } ad-bc=1 \qquad SL(2,\mathbb{C}) \simeq SO(3,1)$$

euclidean conf. gp. in $d=2$

(Lorentzian: $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \simeq SO(2,2)$)

(infinitesimally $f_{global} = \{1, z, z^2\}$) note Schwarzian is zero for these

· one may use a basis of f: z^{n+1}, w/ gen. $l_n = -z^{n+1}\partial_z$ (acting on functions). global: l_{-1}, l_0, l_1 (no central ext).

· composing a series of these infinitesimal transf, under a finite conformal transf transf. as

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial z}{\partial \tilde{z}}\right)^2 \Big[T(z) + \frac{c}{24\pi} \underbrace{\{\tilde{z}, z\}}_{\frac{\tilde{z}'''(z)}{\tilde{z}'} - \frac{3}{2}\left(\frac{\tilde{z}''}{\tilde{z}'}\right)^2} \Big]$$

Schwarzian deriv.

· note invar global part of conf. gp

Physical applications / significance of c

- an immediate application of the Schw is deriving the Casimir eng of the CFT on the cylinder.

  vacuum eng.
  in presence of non-triv. bnd. cond.

- we have $z_{pe} = e^{\frac{w_{cyl}}{R}}$ $\quad w \to \tau + i\sigma$

$$T_{cyl}(w) = \left(z^2 T_{plane} + \frac{c}{2 \cdot 24\pi} \right) \frac{1}{R^2} \qquad \langle T_{pe} \rangle = 0$$

$$H_{cyl} = \underset{\text{Wick rot}}{-} \int d\sigma\, T_{\tau\tau}^{cyl} = - \int d\sigma \left(T_{ww} + \bar{T}_{\bar{w}\bar{w}} \right) = - \frac{2\pi R}{R^2} \frac{c}{48\pi} \cdot 2 = - \frac{c}{12R}$$

- the central charge also enters the algebra of the conserved charges.
  - simplest to work on the cylinder, w/ basis of functions $e^{in(\sigma \pm t)}$ (Lorenzian)
  - conserved charges are $L_n = \int_0^{2\pi R} d\sigma\, \underbrace{T_{t+}}_{T_{++} + \underbrace{T_{-+}}_{0}} e^{in\overbrace{(t+\sigma)}^{x^+}/R}$

$$\delta_m L_n = \int d\sigma\, \delta_m T_{++} e^{inx^+/R} = \int d\sigma\, e^{inx^+/R} \Big[- e^{\frac{imx^+}{R}} \partial_+ T_{++} - \frac{2im}{R} e^{\frac{imx^+}{R}} T_{++}$$

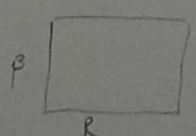
$$+ \frac{c}{24\pi} \left(\frac{im}{R} \right)^3 e^{\frac{imx^+}{R}} \Big] = \int d\sigma\, e^{\frac{i(m+n)x^+}{R}} \left[T_{++} \left(\frac{i}{R}(m+n) - \frac{2im}{R} \right) - \frac{im^3 c}{24\pi R^3} \right]$$

$$= -i(m-n) L_{m+n} - \frac{im^3 c}{12R} \delta_{m+n} = -i [L_m, L_n]$$

$$\Rightarrow \boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n}} \quad \text{Virasoro algebra}$$

shift $L_0^{cyl} = L_0^{pe} - \frac{c}{24}$

- finally the central charge controls the asymptotic density of states



$$Z_{T^2} = \mathrm{Tr}_{\mathcal{H}_R}\, e^{-\beta H} = Z_\beta(R)$$

- switching the interpretation of time & space

$$Z_\beta(2\pi R) = Z_{2\pi R}(\beta) \underset{\text{scale invar}}{=} Z_{4\pi^2 R^2/\beta}(2\pi R)$$

scale invar

$Z_\beta(R) = Z(\beta/R)$

relates low & high-temperature partition f.

- taking $\beta \to 0$

$$Z_\beta(R) = \underbrace{Z_{4\pi^2 R^2/\beta}}_{\infty}(R) \approx e^{-\underbrace{E_0(R)}_{-\frac{c}{12R}}\frac{(2\pi R)^2}{\beta}} = e^{\frac{4\pi^2 R c}{12\beta}} = e^{\frac{\pi^2 c}{3}\frac{R}{\beta}}$$

$$= e^{-\beta F}$$

- thus the free energy is $F(\beta) = -\frac{\pi^2 c}{3}\frac{R}{\beta^2}$ extensive

$$\langle E \rangle = -\partial_\beta \ln Z = \frac{\pi^2 c}{3}\frac{R}{\beta^2} \qquad S = \beta(E - F) = \frac{2\pi^2 c}{3}\frac{R}{\beta}$$

$$\Rightarrow \quad \beta = \sqrt{\frac{\pi^2 c}{3}\frac{R}{E}} \qquad \boxed{S = 2\pi\sqrt{\frac{c}{3} R E}}$$

Cardy's formula