Introduction to holography – Lect XIV

Last time : showed that black hole entropy as computed w/ the Bekenstein–Hawking formula, matches CFT thermal entropy (up to factor of 3/4 in $\mathcal{N}=4$ SYM, & exactly in AdS_3/CFT_2)

- thus, the Bekenstein–Hawking entropy (motivated by analogy b/w the laws of b.h. mech. & the laws of thermodyn) does have a microscopic interpretation as $S = \ln \underset{\sim}{\Omega}$, # microstates (of the CFT dual – whether these microst. can be distinguished in the bulk descr. of the system is still subject to debate)

- many other exact matches of the entropy of (susic) black holes to the entropy of a QM system have been performed. ✓

Obs : the black hole entropy ($A/4G$) that obeys the laws of "b.h. thermodyn" should corresp to the thermodynamic entropy of the system, given its tendency to increase under time evolution

S_{thdyn} : coarse-grained entropy. consider all microscopic configs (states $|\psi\rangle$, or density matrices ρ) consistent w/ the various macroscopic observables we are tracking ($Tr\, \rho A$ = fixed, or $\langle\psi|A|\psi\rangle$ = fixed) & then maximize $S(\rho) = -Tr\, \rho \ln \rho$ over ρ.

This time : it was recently ('06) realised that a notion of fine-grained entropy = entanglement entropy (def in the microsc. CFT descr.) also has a geometric bulk interpretation as the area of a certain surface (in AdS)

spt. geometry $\leftrightarrow$ entanglement structure in dual CFT

spt dynamics (Einst. eqns) $\leftrightarrow$ quantum info-theoretic constraints

Quantum subsystems & entanglement

- in QM, the state of a quantum system is repr. by a state vector $|\psi\rangle \in \mathcal{H}$
  pure state

- more generally, the system can be in a statistical ensemble of $\neq$ state vectors. This situation (char. for sub systems of a quantum system) is characterized by a density matrix

$$\rho = \sum_i \overset{\geq 0}{p_i} |\psi_i\rangle\langle\psi_i|$$

↑ probability that the system be in state $|\psi_i\rangle$

- $\rho = \rho^\dagger$ w/ $\mathrm{Tr}\,\rho = 1$, $\langle \mathcal{O} \rangle = \mathrm{Tr}(\rho \mathcal{O})$

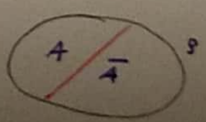
- pure state: $\rho = |\psi\rangle\langle\psi|$ in some basis $\Rightarrow \rho^2 = \rho$ (basis invar); everything else
  mixed

  e.g. Boltzmann: $p_i = e^{-\beta E_i}$

- can distinguish pure from mixed states by computing the von Neumann entropy

$$S = -\mathrm{Tr}\,\rho \ln \rho = -\sum_i p_i \ln p_i$$

- now, consider splitting the system into a subsystem A & its complement $\bar{A}$, $\ni$
  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$



- the reduced density matrix $\rho_A = \mathrm{Tr}_{\bar{A}}\,\rho$ defines the state of the subsystem A.

- the entanglement entropy of A w/ the rest $S_A = -\mathrm{Tr}_A\, \rho_A \ln \rho_A$

- if we assume total state ρ is pure: $\rho = |\psi\rangle\langle\psi|$. Then S_A measures failure of $|\psi\rangle$ to corresp. to a product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_{\bar{A}}\rangle$
  of the form

Example 2 spins

$$\mathcal{H} = \underbrace{\{|\uparrow\rangle, |\downarrow\rangle\}}_{\text{part 1 } (A)} \otimes \underbrace{\{|\uparrow\rangle, |\downarrow\rangle\}}_{\mathcal{H}_{\bar{A}}} = \{ |\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \}$$

taking $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ pure

$$\rho_A = \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi| = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$$

$$S_A = -\mathrm{Tr}_A\, \rho_A \ln \rho_A = -2 \cdot \frac{1}{2} \ln \frac{1}{2} = \ln 2$$ maximally mixed ($n \times n$ $\rho \Rightarrow S_{max} = \ln n$)

- states w/ non-maximal mixing are e.g. $\cos\theta\, |\uparrow\downarrow\rangle + \sin\theta\, |\downarrow\uparrow\rangle$

- can also define the Renyi entropies $\underset{\in \mathbb{Z}}{S_n^A} = \frac{1}{1-n} \ln \mathrm{tr}\, \rho_A^n$ → easier to compute
  - $S_A = \lim_{n \to 1} S_{A,n}$ via analytic continuation in n replica trick

Properties

- entanglement entropy is invariant under unitary time evolution
- subadditivity : A, B subsystems $A \cap B = 0$

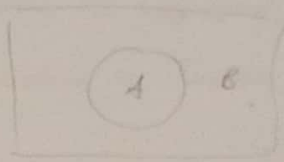
amount info A has about B & vice versa

$$I(A,B) = S(A) + S(B) - S(A \cup B) \geq 0$$ mutual information

  - upper bound on correlations b/w the 2 subsystems $I(A,B) \geq \frac{\left(\langle O_A O_B\rangle - \langle O_A\rangle\langle O_B\rangle\right)^2}{2\, \|O_A\|^2 \, |\langle O_B\rangle|^2}$ bnd. ops in A, B

- also $S(A \cup B) \geq |S(A) - S(B)| \Rightarrow$ if $B = \bar{A}$ & total state is pure then $S(A) = S(\bar{A}) \Rightarrow$ not extensive !
- strong subadditivity : for A, B, C disjoint subsyst. $S(A \cup B) + S(B \cup C) \geq S(B) + S(A \cup B \cup C)$

Entanglement entropy in QFT



- divide space into 2 regions, interested in ρ_A

- def^n problems: $\mathcal{H}_{A\cup B} \neq \mathcal{H}_A \otimes \mathcal{H}_B$ b/c of UV divergences, also for gauge th.

- impose UV cutoff ε, then S_A ok, but is divergent (div. captured by entangl. in the vacuum st.)

  - expect: local integrals over ∂A $\quad \int d^{d-2}x \sqrt{h}\, F[K_{ab}, h_{ab}]$ only even powers

$$S_A^{(div)} \sim a_1 L_A^{d-2} + a_2 L_A^{d-4} + \dots$$

log terms in even d.

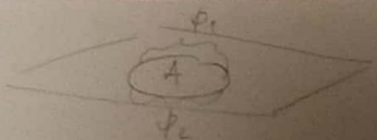
a_1: st. dependent

area div. $\leftrightarrow$ entanglement UV modes across ∂A

  - in a CFT, no scales $\Rightarrow$ $a_i \sim \frac{\#}{\varepsilon^{d-2}}$ etc.

  - coeff L_A^0 (odd d) or $\ln \frac{L_A}{\varepsilon}$ (even d) $\to$ renormalized EE & scales as volume in excited st.

  - interesting b/c coeff of finite / log terms are related to quantities that <u>decrease along RG-flows</u>:



d = 2 $\quad S = \frac{c}{3} \ln \frac{L}{\varepsilon}$ $\quad$ c-thm.

d = 3 $\quad$ finite term EE: $F = -\ln |Z_{S^3}|$

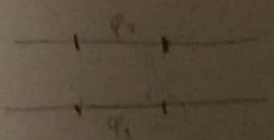
$F_{UV} \geq F_{IR}$

d = 4 $\quad$ coeff. log a

- how to compute EE in QFT

  - ρ_A = path int. over all field configs w/ prescribed bnd. cond. @ A (& some other bnd cond @ e.g. ∞ that encode the state)

  - replica trick

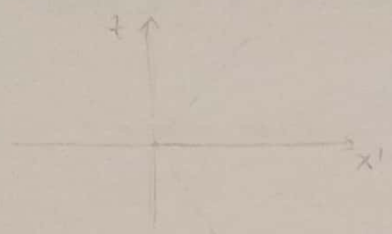


glue to compute $\mathrm{Tr}\, \rho^n$ $\forall n \in \mathbb{N}$ $\to$ Renyi

+ analytically cont. in n + $n \to 1$ $\Rightarrow$ S_A

  - e.g. $S_{CFT_2}(\overset{\ell}{\vdash\!\dashv}) = \frac{c}{3} \ln \frac{\ell}{\varepsilon}$

- a textbook example of QFT entanglement is to consider a half space $x^1 > 0$ in rel. QFT whose domain of dependence is Rindler space



- total state : $|\psi\rangle = |0\rangle_{Minkowski}$
- reduced density matrix $\rho_{Rindler}$ ← thermal

- Bogolyubov transf $a_M = \frac{1}{\sqrt{1-e^{-\beta\omega}}} a_{R_L} - \frac{e^{-\beta\omega/2}}{\sqrt{1-e^{-\beta\omega}}} a^\dagger_{R_R}$ ← Rindler energy

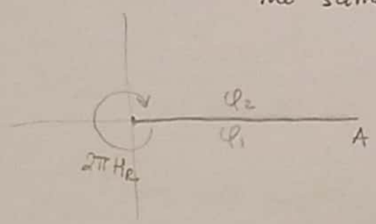
TFD

⇒ Minkowski vacuum can be written as $|0\rangle_M = \frac{1}{\sqrt{Z}} e^{-\frac{\omega\beta}{2} a_L^\dagger a_R^\dagger} |0_{R_L}\rangle|0_{R_R}\rangle$

- tracing over left $Tr_L\, |0_M\rangle\langle 0_M| = \frac{1}{Z} \sum_{n,n'} Tr_L \left(-\frac{\omega\beta}{2}\right)^{n+n'} |n_L\rangle|n_R\rangle\langle n'_R|\langle n'_L|$

$= \frac{1}{Z} \sum_n e^{-\beta \underbrace{E_n}_{\omega n}} |n_R\rangle\langle n_R|$ thermal density matrix

- the same density matrix can be obtained from the path. integral



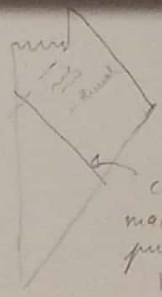
⇒ $\rho = e^{2\pi H_{Rindler}}$ or $\langle \varphi_2|\rho|\varphi_1\rangle$ same as above

- given the analogy b/w Rindler sp & Schwarzschild spt, the Kruskal/infalling vac ↔ $|0_M\rangle$ & Rindler obs ↔ Schwarzschild obs, we can then view $|0\rangle_{inf}$ as an entangled st. for modes outside & inside the horizon

Application to black holes

- definition of generalized entropy $S_{gen} = \frac{A_H}{4G_N} + S_{out}$ ← entanglement entropy of quantum fields outside hor.

! S_{out} : divergent so only sum may be well-def.

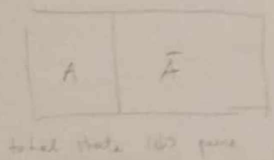


• black hole information problem

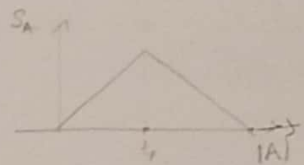
- massless matter in a **pure** quantum state collapses to form a b.h.
- the b.h. then evaporates emitting **thermal** radiation (mixed state)
- eventually, the b.h. disappears & we evolved $|\psi\rangle_{pure} \to \mathcal{S}_{mixed}$, in contradiction w/ unitarity of QM.

• ways out : Hawking may only be trusted until $R_{sch} \gtrsim l_p$ $\to$ remnant (highly entangled w/ rad).
 - info loss $\to\leftarrow$ not supported by AdS/CFT
 - note the calc. of $S_{thermal}$ is a **perturbative** calc. $\to$ could have subtle, non-pert. correlations b/w early & late Hawking quanta that restore unitarity

• the latter is suggested by Page's argument



• if $|A| \ll |\bar{A}|$ then a randomly chosen total pure state $|\psi\rangle$ is likely to look max entangl



$A \sim$ rad
$\bar{A} \sim$ b.h
turnover @ t_{Page}

$S_{BH} \simeq S_{rad}$

- does not support remnant idea

• obtaining the Page curve (entanglement entropy of Hawking radiation) would solve this part of the b.h. info problem

• a more serious issue concerns the fate of an **infalling obs.**
 (reconciling smooth horizon w/ unitarity (s) & EFT @ horizon)
 - modern rephrasing in terms of strong subadditivity of EE:

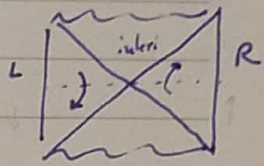
$$\underbrace{S(A\cup B)}_{\simeq 0} + S(B\cup C) \geq S(B) + \underbrace{S(A\cup B\cup C)}_{\simeq S(A)} \qquad \text{if } A, B, C \text{ disjoint}$$

after Page time $S(A\cup B) < S(A)$ (assuming info recovery & unitarity)

- proposed resolutions : firewalls / fuzzballs (structure @ / behind horizon)
 - non-factorization of Hilbert sp on nice slice up until behind horizon (violation of **locality**)

CCA

Eternal black holes & entanglement

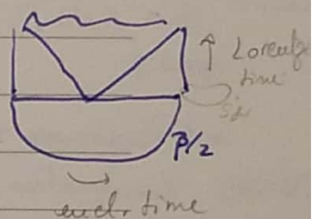


- consider the eternal (large) AdS-Schw. b.h.
  - maximal analytic ext. of Schw-AdS spt. ($\neq$ b.h. collapse)

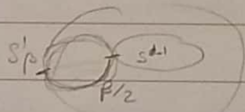
$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-1} + \text{ext.}$$

- 2 bnd $\to$ 2 copies of the dual CFT, state in $\mathcal{H}_L \otimes \mathcal{H}_R$ (identical CFTs)
  (as det. by bnd. cond. near the 2 AdS bnd)

- cutting the Lorentzian geom. @ $t=0$ (time refl.-symm.), we can prepare the state of the quantum fields on the $t=0$ slice via a euclidean path. int, obtained by cutting the eucl. b.h. geom. in half.



This is the Hartle-Hawking constr. of the wavef.



- the question is: what is the corresp. state in the dual $CFT_L \times CFT_R$?

- via AdS/CFT $\quad Z_{\text{bulk}}[\ldots] = Z_{CFT}[\ldots]$
  half-bagel
  $= Z_{CFT}[I_{\beta/2} \times S^{d-1}]$

the claim is that the CFT^2 path int produces the thermofield double st.

$$|TFD\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle_L |n\rangle_R$$

eigenst. of single CFT

w/ prop. that tracing over L/R d.o.f $\Rightarrow$ perf. thermal ρ_R / ρ_L

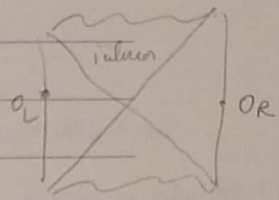
$Z_{CFT}[I_{\beta/2} \times S^{d-1}]$

φ_1 — φ_2, $\beta/2$ $\quad = \langle \varphi_1 | e^{-\frac{\beta H}{2}} | \varphi_2 \rangle = \sum_n \langle \varphi_1 | n \rangle \langle n | \varphi_2 \rangle e^{-\frac{\beta E_n}{2}} = \sum_n \langle \varphi_1 | \ldots | \ldots e^{-\frac{\beta E_n}{2}} | n \rangle \ldots$

$\langle \varphi_2^* | n^* \rangle$

$*$ → reverses direction of time

• thus, we find the rather remarkable fact that, even though the 2 CFTs are entirely non-interacting, $\exists$ a geometric connection b/w them, due to the entanglement in the TFD state.



• this can be probed by e.g. inserting ops.

$$\langle \psi_{TFD} | O_L O_R | \psi_{TFD} \rangle$$

– in CFT, $\neq 0$ due to entanglement $\sim \sum_{n,n'} e^{-\beta\frac{E_n+E_{n'}}{2}} \langle n|O_L|n'\rangle \langle n|O_R|n'\rangle$

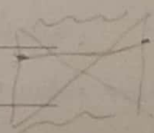
– in gravity, $\neq 0$ b/c $\exists$ a geom conn $\langle O_L O_R\rangle \approx e^{-\Delta \ell_{CR}}$ Δ large.

• this geom. conn. is through the b.h. interior ← emergent due to entanglement. (ER=EPR

• one may use this setup to devise a simpler version of the info paradox.

•

- fix $O_L(0)$ & take $O_R(t)$, t very large. Gravity suggests that $\langle O_L O_R\rangle \sim e^{-\mu\frac{t}{\beta}}$ indefinitely (& we can wait b/c b.h. is eternal)



• however, CFT arg. indicate it becomes $O(e^{-S})$ @ large times

$$\langle O_L O_R\rangle(t) = \frac{1}{Z}\sum_{n,n'} \langle n|O_L|n'\rangle\langle n|O_R|n'\rangle\, e^{-i(E_n-E_{n'})t}\, e^{-\beta\frac{E_n+E_{n'}}{2}} \sim e^{-S}$$

$\sim e^{2S}$ O(1) for $n'=n$

$\sim e^{2S}$ averages to zero for $n\neq n'$ $e^{2S} \sim e^{-S}$

• sharp version of the paradox → can in principle be resolved via non-pert. corr. ($\sim e^{-S}$)

(Malda suggested incl. and AdS saddle, Barbon-Rabinovici argued not enough, for JT SSS argued plateau comes from "D-branes")

Emergent geometry & entanglement

- the literal int. of the TFD constr is that $\sum_n e^{-\beta E_n/2} \, |E_n\rangle \otimes |E_n\rangle =$ 

  $\sum$ disconn. geom $\Rightarrow$ connected

  suggesting entanglement $\overset{\text{rel. to}}{\leftrightarrow}$ connectedness of. spt.

- this can be (semi) sort-of quantitatively motivated by the fact that in this TFD constr, the b.h. entropy $\Leftrightarrow$ entanglement entropy b/w the 2 CFTs

- taking CFT_2 for concreteness, we have $S_{ent} \sim cT$, which decreases as T is decreased

van Raamsdonk
1005.3035
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- also, the mutual info b/w the 2 CFTs $I(A,B) = S(A) + S(B) - S(A \cup B)$

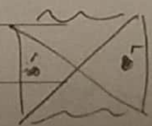
  if I decreases, then corr. b/w the 2 subsystems must decrease, since $I(A,B)$ provides an upper bnd. for them

$$I(A,B) \geq \frac{\left(\langle O_A O_B\rangle - \langle O_A\rangle\langle O_B\rangle\right)^2}{2\,|O_A|^2\,|O_B|^2}$$

these corr. can be seen to become smaller as the temp. of the b.h. is reduced. In Kruskal coord, the BTZ metric reads

$$ds^2 = -\frac{4\,du\,dv}{(1+uv)^2} + 4\pi^2 T^2 \frac{(1-uv)^2}{(1+uv)^2}\, d\varphi^2 \qquad \text{letting } \begin{matrix} u = t+x \\ v = t-x \end{matrix}$$

apparently computing length b/w. 2 sides (@ $t=0$) where φ circumf is $2\pi r$, we find



$$L \sim 2 \ln \frac{\sqrt{\frac{r}{2\pi T}+1} + \sqrt{\frac{r}{2\pi T}-1}}{\sqrt{\frac{r}{2\pi T}+1} - \sqrt{\frac{r}{2\pi T}-1}} \rightarrow \infty \text{ as } T \rightarrow 0$$

so corr $\rightarrow 0$

- so, as T decreases, entanglmt & corr. b/w two sides are reduced.

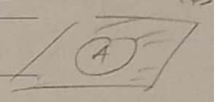
supporting the idea that cls connectivity arises by (cgrav) entanglement.

- relevant for more gen. quest. about how spt. emerges & how it is encoded in geom.

Holographic entanglement entropy

- you saw how, in the AdS/CFT context, the Bekenstein-Hawking formula provides a geometric interpretation for the thermal CFT entropy
  coarse grained (max ρ subject to $\mathrm{Tr}\,\rho A = \mathrm{fixed}$)

- you also learned that, in a local QFT, it is possible to define an entanglement entropy assoc. w/ a spatial subregion A via



$$S(A) = -\mathrm{Tr}\,\rho_A \ln \rho_A \qquad \rho_A = \mathrm{Tr}_{A^c} |\psi\rangle\langle\psi|$$

↑ global state system.

  - fine-grained (much more detailed measure of the state $|\psi\rangle$)

  shows that

  - even in QFT vacuum, $\exists$ lots of entanglement, most coming from correlations just across the entangling surface

$$S_A \sim \# \underbrace{\mathrm{area}(\partial A)}_{\text{univ. div, } \forall \text{ state}} + \# L^{d-4} \quad \& \text{ log terms in even } d.$$

L_A^{d-2} theory-dep.

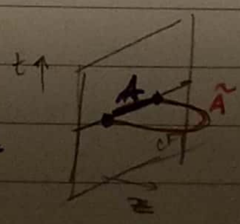
. In highly excited st, esp. $S_A \sim \mathrm{vol}(A)$.

(in CFT_d w/ no other scales $\# \left(\frac{\ell_A}{\varepsilon_{uv}}\right)^{d-2} + \# \left(\frac{\ell_A}{\varepsilon_{uv}}\right)^{d-4} + \ldots$

. coeff. log (even d) $\propto$ central ch, & of 1° (odd d) $\sim$ sphere part. f
(a in 4d)
(monotonous along RG flows)

- $\exists$ a holographic int of S_A?

  - consider hologr. CFT_d (large N, large gap) in a state $|\psi\rangle \leftrightarrow$ smooth classical geom*
  & consider spatial subregion A



RT proposal

$$S_A = \frac{\mathrm{Area}(\tilde{A})}{4G_N}$$

. area cod 2 bulk surface of min area that has same bnd. as A.

* Einstein gravity dual

• in addition, the bulk surface $\tilde{A}$ is homologous to A (can be cont. def into each other in the bulk) (? Check)

• $\tilde{A}$ extremizes area functional. If several sol'ns, take one w/ smallest area

($\exists$ generaliz (HRT) for time dep backgnds, whereby one 1st minimizes area on a bulk spatial slice $\Sigma \supset A$, but then maximizes it over all possible Σ)

• as in the CFT calc, this quantity is divergent (ill def) due to the ∞ volume of AdS $\rightarrow$ structure div. matches CFT (standard UV/IR connection)
regulated by cutoff @ $z=\varepsilon$

. could also consider cutoff-indep quantities, $\ni$ mutual info, $S_A(4S) - S_A(\text{los})$, $\frac{\partial S_L}{\partial L}$ etc

Example : ball-shaped region in CFT vacuum.

$i \in (1, \dots, d-1)$

• dual geom Poincaré AdS $\quad ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + dx^i dx^i\right) = G_{MN}\, dx^M dx^N$

surface lies @ $t=0$

$$\text{Area} = \int d^{d-1}\sigma \sqrt{\det g_{ab}} \qquad g_{ab} = G_{MN} \frac{\partial X^M}{\partial x^a} \frac{\partial X^N}{\partial x^b}$$

· param by x^i & $z(x^i)$

$$g_{ij} = \frac{\ell^2}{z^2(x_i)}\left(\delta_{ij} + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j}\right)$$

$$\sqrt{\det g_{ij}} = \frac{\ell^{d-1}}{z^{d-1}} \sqrt{1 + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^i}}$$

ball $\rightarrow$ spherical symm $dx^i dx^i = dr^2 + r^2 d\Omega^2_{d-2}$, $z = z(r)$ $\quad \text{vol}\, S_{d-2} \int \frac{r^{d-2}\, dr}{z^{d-1}(r)} \sqrt{1+(\partial_r z)^2}$

minimize $\frac{r^{d-2}}{z^{d-1}} \sqrt{1+(\partial_r z)^2}$ w/ soln $z^2 + r^2 = \text{const} = R^2$ (radius ball on the bnd)

• calculate area for e.g. $d=2$ $\quad A = \ell \int \frac{dr}{z} \sqrt{1+(\partial_r z)^2} = \ell\, R \int_0^{R-\eta} \frac{dr}{z^2} \times 2$

where η is such that $\varepsilon^2 + (R-\eta)^2 = R^2$

$\frac{c}{3}$

thus $S_{R} = \frac{\ell R}{4G_3} \cdot 2 \; \frac{1}{R} \, \text{arcth} \frac{r}{R} \Big|_0^{R - \frac{\varepsilon^2}{2R}} = \frac{\ell}{2G_3} \, \underbrace{\text{arcth}\left(1 - \frac{\varepsilon^2}{2R^2}\right)}$

$\approx -\frac{1}{2} \ln\left(\frac{\varepsilon^2}{4R^2}\right)_{L^2}$

$S^{CFT_2}_{L} = +\frac{c}{3} \ln \frac{L}{\varepsilon}$ ← ∝ length

L total length of ~~CFT interval~~

exactly the same as CFT_2 vacuum answer.

omit or leave in

- Special to single interval (univ. answer depending on just anomaly coeff)
  multiple interval does depend on CFT details & only agrees w/ gravity for large c

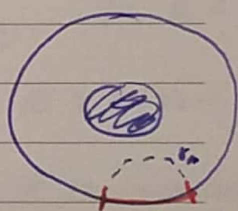
– thus, @ least for $d=2$, we have checked RT yields right answer (single int)

– also check thermal 2d, higher d balls + deformation

Holographic entanglement @ finite temp.

- consider $t=0$ slice BTZ metric

$$dr^2 = \frac{dr^2}{r^2 - 8GM} + r^2 d\varphi^2 \qquad T_H = \frac{\sqrt{8GM}}{2\pi \ell}$$

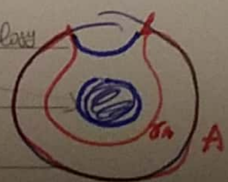


$\in (0, R)$

- $R \ll 2\pi$, S_A computed by length shortest (! ∞, reg!) geod in BTZ geom

$$S_A = \frac{c}{3} \ln\left[\frac{1}{\pi T \varepsilon_{uv}} \, \text{sh}(\pi R T)\right]$$

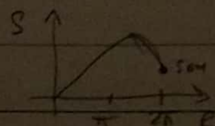
due to homology cond



- for $R \sim 2\pi$, this is not necessarily the shortest geod. (also blue one)

length blue curve $S_A^{blue} = \frac{c}{3} \ln\left[\frac{1}{\pi T \varepsilon_{uv}} \, \text{sh}(\pi(2\pi - R)T)\right] + \underbrace{\frac{2\pi^2}{3} c T}_{\text{b.h. entropy}}$

- must choose $S_A = \min\left[S_A^{red}, S_A^{blue}\right]$



exchange domin @ some $R > \pi$

notice $S_A \neq S_{A^c}$ b/c. of b.h. contr.

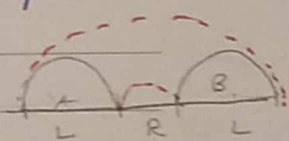
optional

Phase transition in the mutual info

- we saw the Bekenstein - Hawking entropy can undergo phase transitions as a $f(T)$. Similar behaviours can be found in the entanglement entropy

significance

- better work w/ the mutual info $I(A:B) = S(A) + S(B) - S(A \cup B)$ which has the advantage of being finite. Setup:



- 2 regions size L separated by R

$$S(A \cup B)^{\text{alt}} = \frac{c}{3} \ln \frac{R+2L}{\varepsilon} + \frac{c}{3} \ln \frac{R}{\varepsilon} < \frac{c}{3} \ln \frac{L}{\varepsilon} \times 2 \quad \text{for}$$

$R(R+2L) < L^2$, or $R < L(-1+\sqrt{2})$

the mutual info is $I(A:B) = \frac{c}{3} \ln \frac{L^2}{R(R+2L)}$ for $R < L(\sqrt{2}-1)$

0 for $R > L(\sqrt{2}-1)$

- interesting first order phase transition as R is increased.

- applications of this ?

Implications of the holography ent. entropy formula

$S_A = \frac{\text{Area}(\tilde{A})}{4G_N}$, $\forall \tilde{A}$ (over) $\mapsto$ det. in principle the spt. geom. in terms of the entanglmt str. of the CFT

$\mapsto$ highly overconstrained pb, since $\exists$ many more subregion shapes than bulk geom

$\mapsto$ w/ few exceptions (2dCFT) currently hard to underst. why.

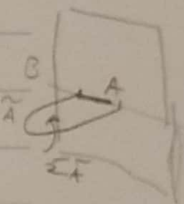
@ large N & strong coupling CFT entang behaves this way

- while RT simply conjectured their formula, note ∃ a bulk derivation for it, i.e. upon assuming that the CFT is descr. by classical Einst. grav. in an a AdS spt.

- the deriv (Malda-Lewcowycz) computes the $Z_{CFT}[B_q]$ using the hologr. descr. that rel. it to $Z_{bulk}[B_q]$ which is extremized by a smooth bulk manif. that ends on B_q @ the bnd.

bnd replica manif to evaluate lhs

$$S_A = \underbrace{\frac{\mathcal{A}(\tilde{A})}{4G_N}}_{O(N^2)} + \underbrace{S_{bulk}(\Sigma_{\tilde{A}})}_{O(1)}$$



-

- additionally, they can gen their prescr. to include quantum corr / $\frac{1}{N}$ corr. that turns out to equal the ent. entropy of bulk fields across the extremal surf.

  ↦ very similar in spirit w/ S_{gen}

  ↦ note that in the above, we first found the min. area surface, & then computed $S_{bulk}(\Sigma_{\tilde{A}})$ across this surface, however, it has been argued one should instead minimize the full S_{gen} → entropy assoc w/ a quantum extremal surface (allows $A_{area} = \langle \hat{\mathcal{A}} \rangle$ to resp. to quantum fluct. of the geom)
  area op
  (relevant @ subleading order)

  ↦ notice this is just a classical surface but whose location is det also by the state of the quantum fields

- this formula will be essential in recovering a unitary Page curve for b.h. evaporation, [illegible].

- entanglement & the emergence of spt.

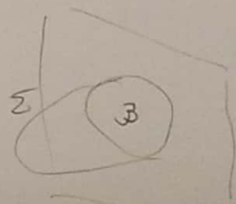
$$= \sum_E e^{-\frac{\beta E}{2}} |E_L\rangle |E_R\rangle \approx \sum_E e^{-\frac{\beta E}{2}} \,|\,\rangle \otimes \langle\,|$$

- holographic entanglement entropy : structure of entanglement must be consistent w/ smooth bulk.

- 1^{st} law of entanglement: defining $\rho = e^{-H_\rho}$ ~ modular Hamiltonian (in general, non-local)

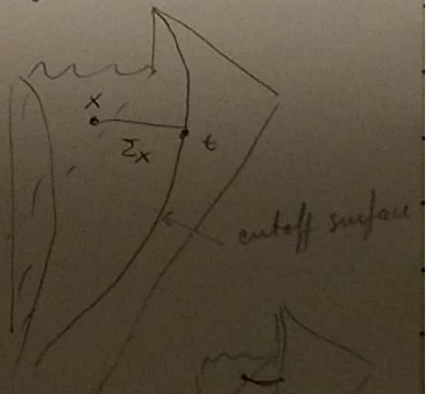
  one can show that $\delta\langle H_\rho\rangle = \delta S_\rho$ under $\rho \to \rho + \delta\rho$ trivial identity

  for a ball-shaped region in a CFT total vacuum $H_\rho = 2\pi \int_{B(R,\vec{x}_0)} d^{d-1}x \, \frac{R^2 - |\vec{x}-\vec{x}_0|^2}{2R} T_{00}(0,\vec{x})$ is local.

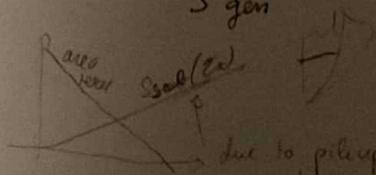


  - assuming that $S_B = \frac{A_\Sigma}{4G}$ (even off-shell)
    & requiring that the 1^{st} law hold $\forall\, \vec{x}_0, R$ (all balls)
    $\Rightarrow$ linearized Einstein eqns ($\in$ q. info consistency constraints)

- quantum extremal surfaces & the Page curve

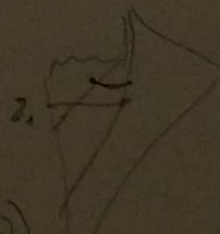


$$S = \min_{\substack{X \\ \uparrow \\ \text{qls. surf.}}} \Big\{ \mathrm{ext}_X \Big(\underbrace{\frac{\mathrm{Area}(X)}{4G_N} + S_{\mathrm{semicl}}(\Sigma_X)}_{S_{gen}} \Big) \Big\}$$



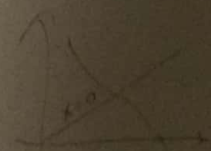
due to pileup of mixed interior modes

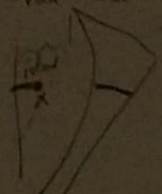
$S_{\mathrm{inside\ cutoff}} \propto S_{BH}$



$S_{rad} = S_{\mathrm{outside\ cutoff}}$

$$S = \min_X \Big\{ \mathrm{ext}_X \Big[\frac{\mathrm{Area}(X)}{4G_N} + S_{\mathrm{semicl}}\big[\Sigma_{rad} \cup \Sigma_{island}\big) \Big\}$$



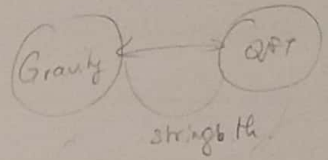


Almheiri, Hartman, Malda et al
2006.06872

Conclusions

Gravity $\xrightarrow{\text{BH thermo}}$ same # d.o.f. as field th. in smaller d } holography

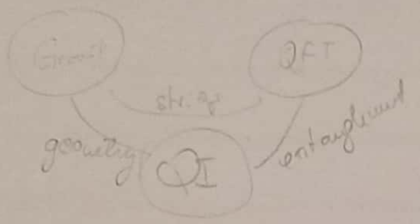
QFT (large N matrix) $\xrightarrow[\text{exp}]{\text{'t Hooft}}$ string th



- very concrete dict AdS/CFT (very part.)

Grav ⟵ QFT

G ⟶ QFT?



in part, gravity $\xrightarrow{\text{Einstein}}$ geometry

entanglement. QI AdS/CFT

Refs. van Raamsdonk 1609.00026

Hartman lect Ch 17-22