Introduction to holography - Lecture XII

- brief review of AdS/CFT aspects we discussed so far

– derivation of the corresp. b/w type IIB string th. on $AdS_5 \times S^5$ and $\mathcal{N}=4$ $SU(N)$ SYM in $d=4$

 - matching of parameters

$\mathcal{N}=4$ SYM	IIB on $AdS_5 \times S^5$
g_{YM}^2	g_s
$\lambda = g_{YM}^2 N$	l_{AdS}^4 / α'^2
N	$\left(\frac{l_{AdS}}{l_P}\right)^4$, or $G_{10} \sim \frac{1}{N^2}$

$\Rightarrow$ $\frac{1}{N}$ corr. in CFT $\Leftrightarrow$ Q.Grav. corr.

$\frac{1}{\sqrt{\lambda}}$ corr. in CFT $\Leftrightarrow$ α' corr.

– when perturbatively expanding the type IIB action around the $AdS_5 \times S^5$ backgnd, interaction terms are suppressed by $1/N$ & $1/\sqrt{\lambda}$ $\Rightarrow$ in the large N limit, fields are approx. free (good starting pt. for the $1/N$ expansion).

– this analysis/param. holds for the specific duality b/w IIB/$AdS_5 \times S^5$ & $\mathcal{N}=4$ SYM, but very similar statements hold for other examples (e.g. for IIB/$AdS_3 \times S^3 \times T^4$ & a 2d CFT known as the D1-D5 CFT, the role of N is played by the CFT central charge, $c = 6 Q_1 Q_5$. The coupling g_{YM} (& θ) get gen. to an 84-dim'l coupling space)

- another important aspect we discussed is the UV/IR connection

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + d\vec{x}^2 + dz^2\right) \qquad z \approx \text{energy scale}$$

$$E_{CFT} \sim \frac{\ell}{z} E_{local}(z) \qquad |\Delta x|_{CFT} = \frac{z}{\ell} |\Delta x|_{local}$$

$\Rightarrow$ large distances in AdS ($z \approx 0$) $\longleftrightarrow$ short dist/high energies in CFT
 IR $\qquad$ UV

→ IR cutoff in AdS ($z = \varepsilon$) $\longleftrightarrow$ UV cutoff in CFT

- an obvious first check of the proposed duality is the matching of symmetries

  - symmetries of the CFT vacuum state $\leftrightarrow$ isometries of AdS ($\times S^{\ldots}$..)

$\mathcal{N}=4$ SYM	IIB / $AdS_5 \times S^5$
conf. gp $SO(4,2)$	isometries AdS_5
global internal $SO(6)$	isometries S^5
(32) fermionic sym	same

D1-D5 CFT	IIB / $AdS_3 \times S^3 \times T^4$
conf. gp $SO(2,2)$	isometries AdS_3
global int. $SO(4) \times U(1)^4$	isometries S^3 & T^4
susy	✓

  - symmetries of the theory $\leftrightarrow$ asymptotic symm of the dual grav. th.

    - look @ a AdS spt w/ appropriate bnd cond for metric & other fields
    - allowed gauge transf. are those diffeos / gauge transf. in the bulk that preserve the bnd. cond.
    - from there, need to discard the trivial diffeos / gauge transf that fall off too fast to contribute to $\forall$ phys cons. charges
    - the quotient $\frac{\text{allowed}}{\text{trivial}}$ = asymptotic symm (large gauge transf.)

  - e.g. asympt. symm. diffeos $AdS_5 \times S^5 \rightarrow \underbrace{SO(4,2)}_{\text{diffeos on AdS}} \times \underbrace{SO(6)}_{\text{on } S^5}$ + gauge symm. from AdS_5 pers.

    – // – $AdS_3 \times S^3 \rightarrow Vir_L \times Vir_R \times \widetilde{SU(2)}_L^{KM} \times \widetilde{SU(2)}_R^{KM}$

  - in general gauge "symm" in the bulk $\leftrightarrow$ global symm in bnd.

    gauge fields in the bulk $\rightarrow$ global cons. currents in the bnd

  - we saw explicitly in the last lecture how the gauge red. of Maxwell on AdS $\rightarrow$ global phase rot on ∂AdS

- we have also discussed the match of the spectrum of pert. of AdS ($\times S^{\circ}$) w/ the CFT a

boundary theory	bulk theory
reps. of the conformal gp $SO(d,2)$	reps. of the isometry gp $SO(d,2)$
conformal local ops (Δ family)	bulk fields $\quad D_{AdS} \leftrightarrow$ conformal (Casimir)
$\Delta(\Delta-d)$ scalars	$m^2\ell^2$ scalars
- vector ops J_μ	vector fields A_M
if conserved	gauge fields
- tensor ops $T_{\mu\nu}$	tensor fields h_{MN}
stress tensor	graviton

many massive vector spin 2 b/c of KK red. from 10d.

- as discussed, the free approx is a good one @ large N $\rightarrow$ GFFs on the bnd.

- will now discuss in detail the dict. for a free massive scalar (neglect int)

The holographic dictionary for a free massive scalar field

$$S = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\,\left(g^{MN}\partial_M\phi\,\partial_N\phi + m^2\phi\right) + \mathcal{O}\left(\tfrac{1}{N}\right)$$

usually obtained via KK red.

- $ds^2 = \ell^2\,\dfrac{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}$ Poincaré coord on AdS_{d+1}, $z \to 0$ bnd.

- the e.o.m. takes the form $\left(z^2\partial_z^2 + z^2\Box_\eta + (1-d)z\partial_z - m^2\ell^2\right)\phi(z,x^\mu) = 0$

- 2^{nd} order eqn $\Rightarrow$ 2 indep. sols. Assuming $\phi \sim z^\Delta f(x^\mu)$ as $z \to 0$, we obt.

$$\Delta(\Delta-d) = m^2\ell^2 \quad \Rightarrow \quad \Delta_\pm = \frac{d}{2} \pm \underbrace{\sqrt{\frac{d^2}{4} + m^2\ell^2}}_{\nu}$$

$\in \mathbb{R}$ if $m^2\ell^2 \geq -\frac{d^2}{4}$ BF bound

$\Delta_+ = \Delta$

$\Delta_- = d - \Delta$

- thus, the near-bnd. exp. of the scalar field takes the form

indep. from p.d.v. of near-bnd analysis, however they get rel via interior.

general sol'n: (general Δ)

$$\phi(z,x^\mu) = \underbrace{\phi_0(x^\mu)\,z^{d-\Delta}}_{\text{non-normalizable}} + \underbrace{\ldots\ldots}_{\text{subleading } (\propto z^{2n}) \text{ terms } \propto \Box^n\phi_0} + \underbrace{\phi_\Delta(x^\mu)\,z^\Delta}_{\text{normalizable}} + \underbrace{\ldots}_{\text{subleading from } \Box\phi_\Delta}$$

(usually) non-normalizable w.r.t. KG norm

∞ expansion

· more precisely $\langle \phi_1, \phi_2 \rangle = -i \int_\Sigma dz\, d^{d-1}x \underbrace{\sqrt{\gamma}}_{\frac{1}{z^d}}\, n^\mu (\phi_1^* \partial_\mu \phi_2 - \phi_2^* \partial_\mu \phi_1) \sim \int \overbrace{z^{2\Delta_\pm}}^{d \pm 2\nu} \frac{dz}{z^{d-1}}$

$\Rightarrow$ z^{Δ_+} is always normalizable

see Klebanov & Witten 9905104

- $z^{\Delta_-} = z^{d-\Delta} \sim z^{\frac{d}{2}-\nu}$ is normalizable if $\nu \in [0,1)$

- if $\nu > 1$, then norm : $\Rightarrow \phi_0 = 0$ $\quad$ (z^{d-D} mode not norm.)
- if $\nu \in (0,1)$, then either $\begin{cases} \phi_0 = 0 \text{ or fixed} & \underline{\text{standard quantiz}} \quad . \ \Delta_+ \subset \text{spectr} \\ \phi_\Delta = 0 \ (\text{fixed}) & \underline{\text{alternate quantiz}} \quad \underline{\Delta_- \subset \text{spectr}} \end{cases}$

how to get ops w/ $\frac{d-2}{2} < \Delta \leq \frac{d}{2}$

- from now on, we will only consider standard quantiz.

• normalizable modes are used to build up the bulk Hilbert sp &, in part, are allowed to fluctuate

• non-normalizable modes are <u>not part</u> of the bulk Hilbert space. They are thus <u>not</u> allowed to <u>fluctuate</u>, but must instead be fixed as boundary cond.

- more precisely, @ the level of the full bulk field we require that

$$\phi(x^\mu, z)\Big|_{z=\varepsilon} = \varepsilon^{d-\Delta} \phi_0(x^\mu) \quad \text{as} \quad \varepsilon \to 0 .$$

"renormalized" bnd. cond. $\quad \phi_0$ finite as $\varepsilon \to 0$

· note that under a rescaling of the CFT coord, $x^\mu \to x'^\mu = \lambda x^\mu$, which in AdS corresp. to the isometry $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, $\phi(x^\mu, z)$ is inert (scalar) $\Rightarrow$

$$\underbrace{\phi_0(x^\mu) = \lambda^{d-\Delta} \phi_0'(x^{\mu\prime})}_{\text{same as source for } \mathcal{O}_\Delta} \qquad \underbrace{\phi_\Delta(x') \lambda^\Delta = \phi_\Delta(x)}_{\text{same transf. as operators of dim } \Delta,\ \mathcal{O}_\Delta}$$

$$\int d^d x \ \phi_0(x^\mu)\, \mathcal{O}_\Delta(x^\mu)$$

- note that $\Delta = d$ (marginal op) $\Leftrightarrow$ $m=0$ (bulk field)

  $\Delta < d$ (relevant op) $\Leftrightarrow$ $m^2 < 0$, & for $\Delta < \frac{d}{2}$ they need to be realised via alternate quantiz

  $\Delta > d$ irrelevant $\Leftrightarrow$ $m^2 > 0$.

- the source terms are usually turned on $\frac{1}{\infty}$ to compute corr. f.
  - for relevant ops, can also turn on the sourced a finite amount (fixed)

    $\Rightarrow$ holographic RG flows

- we are now ready to state the AdS/CFT dict. as an equality of part. f.

$$\boxed{Z_{AdS_{d+1}}\left[\phi(z,x^\mu)\big|_{z\to 0} = z^{d-\Delta}\phi_0\right] = e^{W_{CFT}[\phi_0]} = \left\langle e^{\int d^dx\, \phi_0(x)\mathcal{O}(x)}\right\rangle_{CFT}}$$

bulk partition f. w/ the prescribed bnd. cond. for the fields.

generating f'l of connected corr. f. in the CFT.

- arbitrary corr. f. can be obtained by differentiating $W_{CFT}[\phi_0(x)]$ w.r.t. $\phi_0(x)$

- the above dict. is true in general (bulk = string. th.) but we will concentrate on the sugra limit in which $Z_{AdS}[\phi_0] \approx e^{-S_{sugra}[\phi_0]}$

  $N\to\infty$, λ large

  saddle pt

  bulk on-shell action w/ bnd. cond ϕ_0

- both sides are in fact divergent : Z_{AdS} [IR] $\Leftrightarrow$ Z_{CFT} [UV]

  $\Rightarrow$ need to renormalize in order to obtain a finite answer holographic renorm.

- therefore, we first renormalize, then take functional deriv. w.r.t. ϕ_0

- will show how this works in practice for the 2pf.

(include Petrica lecture notes)

- thus, more concretely, in this case the AdS/CFT dict. can be written as

$$Z_{bulk}\left[\phi(z,x^\mu)\big|_{z\to 0} = z^{d-\Delta}\phi_0(x)\right] = \left\langle e^{-\int d^dx\, \phi_0(x)\, O_\Delta(x)} \right\rangle_{CFT}$$

generating f. of gauge-invar ops. in the CFT $e^{-W_{CFT}[\phi_0]}$

- arbitrary corr. f. can be obtained by differentiating $W_{CFT}[\phi_0(x)]$ w.r.t. $\phi_0(x)$
- true in general (bulk ~ string th) but we will concentrate on the sugra limit $N\to\infty$, λ large, in which $Z_{bulk}[\phi_0] \approx e^{-S_{sugra}[\phi_0]}$

saddle pt — bulk on-shell action w/ bnd. cond ϕ_0

6 Two-point function

- only quadratic part of the action needed $S = \int d^{d+1}x \sqrt{g}\, \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right)$
- the soln is as before $\phi(z,x) = \phi_0 z^{d-\Delta} + \dots \quad + \phi_\Delta z^\Delta + \dots$
- $\phi_0(x^\mu)$, $\phi_\Delta(x^\mu)$ are indep from the p.d.v of the near bnd. analysis; however, they get related by requiring that the soln not blow up ~~in~~ as $z\to\infty$
- solving the wave eqn in momentum sp $\phi(z,x) = e^{ip_\mu x^\mu} f(z)$ in Eucl. th

sols $\quad z^{\frac{d}{2}} I_\nu(pz)$ & $z^{\frac{d}{2}} K_\nu(pz) = z^{\frac{d}{2}}\left(I_\nu(pz) - I_{-\nu}(pz)\right)$

$p = |p|$ $\quad I_\nu = \left(\frac{pz}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$ $\quad = z^{\frac{d}{2}+\nu}\left(\frac{p}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} - z^{\frac{d}{2}-\nu}\left(\frac{p}{2}\right)^{-\nu}\frac{1}{\Gamma(1-\nu)}$

$\Rightarrow \phi_\Delta = -\left(\frac{p}{2}\right)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(\nu+1)}\phi_0$

- let us now evaluate the on-shell action
  - rather $\delta S_{on\text{-}shell}[\phi_0] \leftrightarrow \langle O_\Delta(x)\rangle_{\phi_0}\, \delta\phi_0$

$$\delta S = \int d^{d+1}x \sqrt{g}\,\underbrace{(m^2\phi - \Box\phi)}_{0}\,\delta\phi + \int d^dx \sqrt{\gamma}\, \underbrace{n^\mu}_{\text{normal to the bnd @ } z=\epsilon}\partial_\mu\phi\, \delta\phi$$

$-\frac{z}{\epsilon}\partial_z$

$\underbrace{\epsilon}_{\text{regulator}}$ IR/UV

$$\delta S_{on\text{-}sh} = -\int d^dx\ \frac{z}{\ell}\,\partial_z\phi\,\delta\phi\cdot\left(\frac{\ell}{z}\right)^d = -\int_{z=\varepsilon} d^dx \left(\frac{\ell}{z}\right)^{d-1}\Big[\ \phi_0(d-\Delta)\,z^{d-\Delta-1} + \dots$$

$$\dots + \Delta\phi_\Delta\,z^{\Delta-1} + \dots\Big]\cdot\Big(\ \delta\phi_0\ z^{d-\Delta} + \dots\ +\ \delta\phi_\Delta z^0 + \dots\Big)$$

Note:

- coeff $\phi_0\delta\phi_0$ divergent $\dfrac{\varepsilon^{2d-2\Delta-1}}{\varepsilon^{d-1}} = \varepsilon^{d-2\Delta}$ div for $\Delta > \frac{d}{2}$
- subleading divs $\Box\phi_0\,\delta\partial^2\phi_0$
- finite term (ε^0) $\quad -\big(\ \phi_0(d-\Delta)\,\delta\phi_\Delta + \Delta\phi_\Delta\delta\phi_0\big)$

does not have a good variational principle, as only ϕ_0 should be held fixed (so, $\delta S_{onsh} \propto \delta\phi_0$)

- to obtain meaningful & finite results we should - as in QFT. add bnd. counterterms to remove the divs. In AdS/CFT, the counterterms are local, Lorentz invar, & depend only

7 intrinsically on the bnd geometry ($\phi|_\varepsilon$, $\gamma|_\varepsilon$ etc.).

$\rightarrow$ holographic renormalization

- easy to see the leading divergence can be absorbed via the counterterm $\quad S_{ct} = \frac{c}{\ell}\int d^dx\,\sqrt{\gamma}\ \phi^2(x,z)\big|_{z=\varepsilon}$ choosing c appropr.

$$\delta S_{ct} = \frac{2c}{\ell}\int_{z=\varepsilon} d^dx\ \left(\frac{\ell}{z}\right)^d\big(\phi_0\ z^{d-\Delta} + \dots\ \big)\big(\delta\phi_0\ z^{d-\Delta} + \dots\ \big)$$

$$-\phi_0\delta\phi_0(d-\Delta) + 2c\,\phi_0\delta\phi_0 = 0 \quad \Rightarrow \quad c = \frac{d-\Delta}{2}$$

- this ct. also modifies the coeff of the subleading terms (div. or not) & in part, the finite term coeff. becomes

$$-\big(\phi_0(\cancel{d}-\Delta)\,\delta\phi_\Delta + \Delta\phi_\Delta\delta\phi_0\big) + (d-\Delta)\big(\cancel{\phi_0}\delta\phi_\Delta + \phi_\Delta\delta\phi_0\big)$$

$$= (d-2\Delta)\,\phi_\Delta\delta\phi_0$$ problematic term cancels! & $\delta S \propto \delta\phi_0$

- the subleading divs can be cancelled by ct $\int d^dx\sqrt{\gamma}\ \phi\,\Box^n\phi$ which do not affect the finite term.

$\cdot$ Thus $\dfrac{\delta S_{\text{on-shell}}[\phi_0]}{\delta\phi_0} = (2\Delta-d)\,\phi_\Delta \overset{\sim\, \epsilon^{\Delta-1}}{} = \langle O_\Delta\rangle_{\phi_0}$

$\Rightarrow$ the coeff ϕ_Δ in the near-bnd. expansion of the scalar repr. the expectation value of the dual bnd. operator

$\cdot$ to obtain the 2pt.f $\langle O_\Delta O_\Delta\rangle = \dfrac{\delta\langle O_\Delta\rangle}{\delta\phi_0} \propto p^{2\nu}$ in momentum space $\langle O(p)\,O(-p)\rangle$

$\cdot$ Fourier-transforming $\int d^dp\; e^{ip\cdot x}\, p^{2\nu} \sim \dfrac{1}{(x_1-x_2)^{2\Delta}}$ $\quad \Delta = \frac{d}{2}+\nu$

exactly the 2p.f of an operator of dim. Δ.

Comments

- here, free scalar ; higher spin fields very similar $\to$ higher spin ops in CFT
- for massless gauge fields in the bulk $(A_\mu, g_{\mu\nu}) \to$ cons. currents on bnd ; cons. eqn. obtained from near-bnd.-analysis (hologr. Ward identities, including anomalies)
- interesting to note $\dfrac{1}{|x-y|^{2\Delta}}$ is $e^{-\Delta\ell}$ $\quad \ell$ ren. geod. length b/w the pts x, y on the bnd

also holds more gen. in non-vac. spt.

Higher-point functions

- need to include interaction terms in the bulk. e.g. for $\frac{\lambda \phi^{n+1}}{n+1}$, the modified e.o.m is

$$(\Box - m^2)\phi = \lambda \phi^n$$

- since λ is small, the sol'n can be expanded perturbatively as

$$\phi = \phi^{[0]} + \lambda \phi^{[1]} + \lambda^2 \phi^{[2]} + \dots$$

where $(\Box - m^2)\phi^{[0]} = 0$ $\quad (\Box - m^2)\phi^{[1]} = \lambda \phi^n_{[0]}$ etc.

therefore $\phi^{(1)}(y) = \int d^{d+1}y' \sqrt{g}\, \underbrace{G_{bb}(y|y')}_{} \phi^n_{[0]}(y')$

bulk-to-bulk propagator $(\Box_y - m^2) G_{bb}(y|y') = \frac{1}{\sqrt{g}} \delta^{d+1}_{(y,y')}$

- the zeroth order sol'n $\phi_{[0]}$ is itself det. by the bnd. value ϕ_0, of the field using the bulk-to-bnd propagator $K_{b\partial}(y|x)$

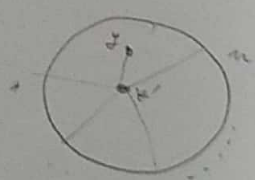
$$(\Box_y - m^2) K_{b\partial}(y|x) = 0 \qquad \lim_{z\to 0} K_{b\partial}(z, x'|x) \sim z^{d-\Delta} \delta^{(d)}(x-x')$$

$$\phi_{[0]}(y) = \int d^d x \; K_{b\partial}(y|x)\, \phi_0(x)$$

sol'n unique euclidean

the sol'n turns out to be (Poincaré coord) $\quad K_{b\partial} = \left(\frac{z}{z^2 + (x-x')^2} \right)^{\Delta}$

- thus, the perturbative bulk field can be constr. via a Witten diagram

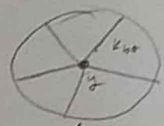


- the answer for the $\overset{n+1}{\vee}$ corr. f^n is given by extracting the coeff. of the z^Δ term from $\phi^{(1)}$ (= 1pf in presence of the source) & differentiating n times

- since $\lim_{z\to 0} G_{bb}(z,x|z',x') = \frac{z^\Delta}{2\nu} K(z|z',x')$, the final answer for the n+1-pf is

· the end result for the n+1 - point function is

$$\langle O(x_1)\dots O(x_n)\rangle = -\frac{\delta S^{ren}_{ds}}{\delta\phi_{(0)}(x_1)\dots\delta\phi_{(0)}(x^n)} = \lambda\int d^{d+1}y\; K_{bd}(x_1,y)\dots K_{bd}(x_{n+1},y)$$



· more generally, CFT corr. f. can be computed via AdS diag. w/ pts ending on the bnd.

· ∃ an alternate way of computing CFT corr as bnd limits of bulk ones.

$$\langle O_1(x_1)\dots O_n(x_n)\rangle_{CFT} = \lim_{z_i\to 0}\prod_i 2\nu_i z_i^{-\Delta_i}\langle\phi_1(z_1,x_1)\dots\phi_n(z_n,x_n)\rangle$$