Introduction to holography – Lecture VII

Last time : derivation of a particular instance of the AdS/CFT corresp. from string theory, (namely b/w $\mathcal{N}=4$ SU(N) SYM & type IIB string th. on $AdS_5 \times S^5$), strong-weak, hard to check.

Next few times : general introduction to CFT_d's & some of their (generic) properties (symmetries, operators, str. of corr. f, etc) in view of a structural match w/ prop. of gravitational th. in AdS_{d+1} space-times

CFTs : relevant in many areas of physics where the behaviour of the system becomes scale-invar:

- continuous (2^{nd} order) phase transitions & quantum critical points
- IR behaviour in QFT (appearance in AdS/CFT)
- UV –||–
- worldsheet string theory (CFT_2)

· oftentimes, theories invariant under scaling are also invar. under the larger gp of conformal transformations : coordinate transf that leave the metric invar. up to a (possibly space-time dependent) overall rescaling

· concretely, under $x^\mu \to x^\mu + \xi^\mu$, $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \propto g_{\mu\nu}$ =>

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \frac{2}{d} g_{\mu\nu} \nabla_\lambda \xi^\lambda \qquad \text{conformal Killing vect.}$$

· let us fix the metric to be Minkowski : $\eta_{\mu\nu}$ (Lorentzian signature) or $\delta_{\mu\nu}$ (Euclidean)

· denoting the RHS as $f(x) = \frac{2}{d} \nabla_\lambda \xi^\lambda$, we have $2 \nabla_\mu \nabla_\nu \xi_\rho = g_{\mu\rho} \nabla_\nu f + g_{\nu\rho} \nabla_\mu f - g_{\mu\nu} \nabla_\rho f$

$$\Rightarrow \quad d \, \nabla_\mu \nabla_\nu f = 2 \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f$$

· 2 cases: $d>2$. then $(d-2)\nabla_\mu\nabla_\nu f = -g_{\mu\nu}\Box f$, w/ sol'n $f(x) = 2\lambda + 4 b_\mu x^\mu = \frac{2}{d}\partial_\lambda \xi^\lambda$

antisym

$$\Rightarrow \xi^\mu = \underbrace{a^\mu}_{\substack{\text{translations} \\ d}} + \underbrace{\omega^{\mu\nu} x_\nu}_{\substack{\text{Lorentz (L)} \\ \frac{d(d-1)}{2} \\ \text{rotation (E)}}} + \underbrace{\lambda x^\mu}_{\substack{\text{scale} \\ 1}} - \underbrace{b_\mu x^2 - 2x_\mu b_\nu x^\nu}_{\substack{\text{special conformal} \\ d}}$$

– total of $\frac{1}{2}(d+1)(d+2)$ transf.

· if $d=2$, then we simply have $\Box f = 0$ $\partial_z\partial_{\bar z} f = 0$ (Eucl. complex coord)
$\partial_+\partial_- f = 0$ (Lorentzian null).

the general sol'n is $f = f(z) + \bar f(\bar z)$ ∞ dim'l
x^+ x^-

· finite conformal transf : $\omega_{\mu\nu} \to$ Lorentz $\Lambda^T\eta\Lambda = \eta$ or rotations $M^TM = 1$

b_μ : special conformal transf.

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

· this can be understood as the composition of an inversion $\left(x^\mu \to x'^\mu = \frac{x^\mu}{x^2}\right)$

+ translation + another inversion

$$x^\mu \to \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2} = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$$

· note the SC transf. is continuously conn. to the identity, whereas the inversion is not. (the CFT need not have it as a symm.)
(so this) maps it to parity

· can think of SC as a transf. that fixes the origin ($x^\mu = 0$) & moves ∞, in contradistinction w/ transl, which fix ∞ & move the origin

· all in all, a finite conf. transf. takes the form $\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x)\Lambda^\mu{}_\nu(x)$

position-dep. rescalings & rotations.

$(x'-y')^2 = \Omega(x)\Omega(y)(x-y)^2$

(angle-preserving transf.).

- the conformal algebra is $SO(d,2)$ (Lorentzian), or $SO(d+1,1)$ in eucl.

(Slavnov 9712074)

Poincaré $\begin{cases} [M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \end{cases}$ $\quad [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho}$

Lie bracket vector fields

$-x^2\partial_1$

K_μ vector ? SC $\begin{cases} [M_{\mu\nu}, K_\rho] = \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu \end{cases}$ $\quad [D, P_\mu] = P_\mu \quad [D, K_\mu] = -K_\mu$

$(K_\mu)^\delta = -2x^\delta x^\mu + x^2\eta^{\mu\delta}$

raising / lowering for D eigenval

$$[K_\mu, P_\nu] = -2\eta_{\mu\nu} D + 2 M_{\mu\nu}$$

$\mu \in \{0, \dots, d-1\}$

standard $[\ ,\ ] = -i[\ ,\ ]_{LB}$. (multiply all gen by $-i$)

- defining $J_{\mu\nu} = M_{\mu\nu}$ $\quad J_{-1,d} = D$ $\quad J_{\mu,-1} = \frac{1}{2}(P_\mu + K_\mu)$ $\quad J_{\mu d} = \frac{1}{2}(P_\mu - K_\mu)$

  they satisfy $SO(d,2)$, w/ the $(-1$ directions $(-1,0)$. (also works in eucl) w/ slightly ≠ signs)

- quad. Casimir $C_2 = \frac{1}{2} J^{AB} J_{AB}$. (also quartic)

- interesting to define: $M'_{pq} = J_{pq}$ $\quad p,q \in \{1, \dots, d\}$

  $D' = i J_{-10}$ $\quad P'_p = J_{p,-1} + i J_{p,0}$ $\quad K'_p = J_{p,-1} - i J_{p,0}$

  these new generators satisfy the $SO(d+1,1)$ alg of the euclidean conformal gp.

  however, their hermiticity properties are $M'^\dagger = M'$, $D'^\dagger = -D'$

  $P'^\dagger = K'$, $K'^\dagger = P'$

- unitary reps are ∞-dim'l, char. by reps of its maximal compact subgp $SO(2) \times SO(d)$ (gen by $J_{0,-1} \sim P_0 + K_0$ & J_{pq})

- would like to underst. how the conformal gen. act on local operators O_α

$$O_\alpha(x) = e^{i x^\mu P_\mu} O_\alpha(0)\, e^{-i x^\mu P_\mu}$$

true in any quantity of K, M

- the action of the conf. gen on $O_\alpha(x)$ is entirely det. by their action on $O_\alpha(0)$

$$\underset{\substack{\uparrow \\ \text{generators}}}{[G, O_\alpha(x)]} = e^{i x^\mu P_\mu} \Big[\overbrace{e^{-i x^\mu P_\mu} G\, e^{i x^\mu P_\mu}}^{\tilde{G}}, O_\alpha(0) \Big] e^{-i x^\mu P_\mu}$$

$$G - i x^\mu [P_\mu, G] + \frac{i^2}{2} x^\mu x^\nu [P_\mu [P_\nu, G]] + \dots$$

- luckily, these nested commutators terminate

$$\tilde{D} = D + x^\mu P_\mu \qquad \tilde{M}_{\mu\nu} = M_{\mu\nu} + x_\nu P_\mu - x_\mu P_\nu$$

$$\tilde{K}_\mu = K_\mu - 2 x_\mu D + 2 M_{\mu\rho} x^\rho - 2 x_\mu x^\lambda P_\lambda + x^2 P_\mu$$

$\Rightarrow$ the action of conf. gen on $\mathcal{O}_\alpha(x)$ is determined by the action of $D, M_{\mu\nu}, K_\mu$ (stabilizer$^{x=0}$) on $\mathcal{O}_\alpha(0)$.

$$[D, \mathcal{O}_\alpha(0)] = \underbrace{\Delta^\alpha{}_\beta}_{-i\Delta} \mathcal{O}_\beta(0) \qquad [M_{\mu\nu}, \mathcal{O}_\alpha(0)] = (S_{\mu\nu})_\alpha{}^\beta \mathcal{O}_\beta(0) \qquad [K_\mu, \mathcal{O}_\alpha(0)] = (\mathcal{k}_\mu)^\gamma{}_\beta \mathcal{O}_\beta(0)$$

– natural to diagonalize dilatations $\mapsto$ scaling ops $e^{i\alpha D} \mathcal{O}(0) e^{-i\alpha D} = e^{+\alpha\Delta} \mathcal{O}(0)$

$\mathcal{O}'(x') = \lambda^{-\Delta} \mathcal{O}(x) \quad x' = \lambda x \qquad iD = -x\partial_x \quad (e^{-\alpha})^{(-\Delta)}$

– since K_μ acts as a lowering op. for D, natural to look @ ops annih. by it

$\Rightarrow$ primary operators $\quad [K_\mu, \mathcal{O}(0)] = 0$

. if we act w/ P_μ $\Rightarrow$ descendants (∞ # of these).

- particularizing to such $\underset{\text{primary}}{\text{ops.}}$ we have

$$[P_\mu, \mathcal{O}(x)] = -i \partial_\mu \mathcal{O}(x) \qquad [D, \mathcal{O}(x)] = -i\Delta \mathcal{O}(x) - i x^\mu \partial_\mu \mathcal{O}(x)$$

$$[K_\mu, \mathcal{O}(x)] = 2i\Delta \mathcal{O}(x) x_\mu + 2 x^\rho (S_{\mu\rho} \mathcal{O}(x)) - 2 x_\mu x^\lambda (-i\partial_\lambda \mathcal{O}(x)) + x^2(-i\partial_\mu \mathcal{O}(x))$$

$$[M_{\mu\nu}, \mathcal{O}(x)] = (S_{\mu\nu} \mathcal{O}(x)) - i x_\nu \partial_\mu \mathcal{O}(x) + i x_\mu \partial_\nu \mathcal{O}(x)$$

- these impose stringent constraints on the corr. f. of primary ops. Using

$$0 = \langle 0 | [G, \mathcal{O}_1 \ldots \mathcal{O}_n] | 0 \rangle = \sum_i \langle 0 | \mathcal{O}_1 \ldots [G, \mathcal{O}_i] \ldots \mathcal{O}_n | 0 \rangle$$ in some quantity (cond.).

we have, e.g. for a scalar 2p.f. $\langle \mathcal{O}(x) \mathcal{O}'(y) \rangle = f(x-y)$

$$(\Delta_1 + \Delta_2) f + (x^\mu \partial_{x^\mu} + y^\mu \partial_{y^\mu}) f = 0 \quad \Rightarrow \quad f = \frac{1}{|x-y|^{\Delta_1 + \Delta_2}} \qquad \Leftrightarrow \quad \underline{\Delta_1 = \Delta_2}$$

$$2\Delta_1 x^\mu + 2\Delta_2 y^\mu + 2 x_\mu x^\lambda \partial_{x^\lambda} f - x^2 \partial_{x^\mu} f + 2 y_\mu y^\lambda \partial_{y^\lambda} f - y^2 \partial_{y^\mu} f = \frac{(\Delta_1 - \Delta_2)(x^\mu - y^\mu)}{|x-y|^{\Delta_1 + \Delta_2}}$$

later

- for an operator w/ spin (e.g. spin 1), we can repr. $(S_{\mu\nu})^a{}_b = -i(\delta_\mu{}^a \eta_{\nu b} - \delta_\nu{}^a \eta_{\mu b})$
(non-unitary, finite-dim'l rep. in Lorentzian) (but ok in eucl.)

e.g. the Lorentz ${}^{1st.}$ constraint on the corr. f. $\langle O^\rho(x) O^\sigma(y)\rangle$ is

$$-i\,\delta_\mu{}^\rho f_\nu{}^\sigma + i\,\delta_\nu{}^\rho f_\mu{}^\sigma - i x_\nu \partial_\mu f^{\rho\sigma} + i x_\mu \partial_\nu f^{\rho\sigma}$$
$$- i\,\delta_\mu{}^\sigma f^\rho{}_\nu + i\delta_\nu{}^\sigma f^\rho{}_\mu - i y_\nu \partial_{y\mu} f^{\rho\sigma} + i y_\mu \partial_{y\nu} f^{\rho\sigma} = 0$$

taking $f^{\rho\sigma} = \dfrac{\eta^{\sigma\rho} + \alpha \dfrac{(x^\rho-y^\rho)(x^\sigma-y^\sigma)}{(x-y)^2}}{(x-y)^{\Delta_1+\Delta_2}}$ each term satisfies eqn. above (check!)

- to fully fix the corr. f, need to also solve the special conformal Ward id. exercise

- the above infinitesimal transf of primary ops under conformal transf. result in the following finite transf $\mathcal{U} O^\alpha(x)\,\mathcal{U}^{-1} = \Omega(x')^\Delta D(R(x'))^\alpha{}_\beta O^\beta(x')$
where the finite conf. transf is $\dfrac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) R^\mu{}_\nu(x)$ (or $\Omega(x') R^\mu{}_\nu(x')$)

$$\Rightarrow \langle O_1^{\alpha_1}(x_1) \ldots O_n^{\alpha_n}(x_n)\rangle = \Omega(x_1')^{\Delta_1} \ldots \Omega(x_n')^{\Delta_n} \langle D(R(x_1'))^{\alpha_1}{}_{\beta_1} O^{\beta_1}(x_1') \ldots D(R(x_n'))^{\alpha_n}{}_{\beta_n} O^{\beta_n}_{n}(x_n')\rangle$$

- for example, the 2-pct. f. of scalar ops must transform as

$$\langle O_1(x_1) O_2(x_2)\rangle = \Omega^{\Delta_1}(x_1')\, \Omega^{\Delta_2}(x_2') \langle O_1(x_1') O_2(x_2')\rangle \quad *$$

- trivial to see $\dfrac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}}$ transf. correctly under transl. $(\Omega=1)$
rotations $(\Omega=1)$, and dilatations $\Omega=\lambda$ $(x'^\mu = \lambda x^\mu)$

- to check the special conformal transf, sufficient to check transf. under inversions $x^\mu \to \dfrac{x^\mu}{x^2}$ w/ $\Omega = \dfrac{1}{x^2} = x'^2$. Note that, under an inversion
$(x_1-x_2)^2 = \dfrac{(x_1'-x_2')^2}{\Omega(x_1)\Omega(x_2)}$

- then $\dfrac{1}{|x_{12}|^{\Delta_1+\Delta_2}} = \Omega(x_1')^{\frac{\Delta_1+\Delta_2}{2}} \Omega(x_2')^{\frac{\Delta_1+\Delta_2}{2}} \dfrac{1}{|x_{12}'|^{\Delta_1+\Delta_2}}$ same as * if $\Delta_1=\Delta_2$

- 3-point functions of scalar ops are similarly entirely fixed by conf. symm, up to an overall 3p.f. coefficient

$$\langle O_1(x_1) O_2(x_2) O_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{31}|^{\Delta_1+\Delta_3-\Delta_2}}$$

meaningful after norm. the 2pf. to δ_{ij}

- 4-point functions are not fixed by conformal symm b/c one can construct conformal invar. w/ 4 points (from n pcts, will have $\frac{n(n-3)}{2}$ invars).

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} \qquad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}$$

- use a transl. to place $x_1 = (0, \dots, 0)$
- use a rotation/boost* to set $x_2 = (0, \dots, 0, 1)$ *assume spacelike sep.
- use an orthogonal rot to place $x_3 = (0, \dots 0, x_3^{d-1}, x_3^d)$
- use a special conf. transf. to send x_3 to the line joining x_1 & x_2

$$x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \Rightarrow x_1'^\mu = x_1^\mu = (0, \dots, 0)$$

$$x_2'^\mu = \frac{x_2^\mu - b^\mu}{(x_2 - b)^2} \qquad x_3'^\mu = \frac{x_3^\mu - b^\mu x_3^2}{1 - 2b\cdot x_3 + b^2 x_3^2}$$

choosing $b^\mu = b\,\delta^{\mu, d-1}$, we have $-b = \frac{x_3^{d-1} - b\, x_3^2}{x_3^d}$, or $b = \frac{x_3^{d-1}}{x_3^2 - x_3^d}$ $\left(x_3^2 = (x_3^{d-1})^2 + (x_3^d)^2\right)$

so, all 3 points are in a line

+ rot. + dilatation, can set

$$x_1'' = (0, \dots, 0) \qquad x_2'' = (0, \dots, 0, 1) \qquad x_3'' = (0, \dots, x_3^{''d})$$

(an additional SC w/ $b' = (0, \dots, -x_3^{''d})$ places x_3''' @ ∞, while leaving x_2''' on the same line.

- a 4^{th} pct would be lying in the same plane (2d) as other 3. label as $(0, \dots, z, \bar{z})$ $x^{d-1} + i x^d$

$P_1 = (0, \dots, 0)$, $P_2 = (0, \dots, z, \bar{z})$, $P_3 = (0, \dots, 1)$ label this as P_2

$P_4 = (0, \dots, \infty)$

then $u = z\bar{z}$ $v = (1-z)(1-\bar{z})$

- the general form of the 4-pct-f is $\langle O_1 \dots O_4\rangle = \frac{A(u,v)}{(x_{12}^2\, x_{34}^2)^{\Delta}}$ same Δ.