# Nonlinear behavior and turbulence spectra of drift waves and Rossby waves

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Spectrum cascade in drift wave turbulence in a magnetized plasma as well as Rossby wave turbulence in an atmospheric pressure system are studied based on a three-wave decay process derivable from the model equation applicable to both cases. The decay in the three-way interaction occurs to smaller and larger values of $|k|$. In a region of large wavenumbers this leads to the dual cascade; the energy spectrum cascades to smaller $|k|$ and the enstrophy spectrum to larger $|k|$, similar to the case of two-dimensional Navier–Stokes turbulence. In a small wavenumber region a resonant three-wave decay process dominates the cascade process, and an anisotropic spectrum develops. As a consequence of the cascade, zonal flows in the direction perpendicular to the direction of inhomogeneity appear which presents a potential implication for the particle confinement in a turbulent plasma.

## I. INTRODUCTION

A unique property of two-dimensional hydrodynamic turbulence is that energy cascades into longer wavelengths; that is, it has the so-called inverse cascade process.[1-4] Electrostatic turbulence in $\mathbf{E}\times\mathbf{B}$ guiding center plasma obeys the Euler equation for an incompressible fluid; hence, it possesses the same cascading property as two-dimensional hydrodynamic turbulence.[5,6]

In weak turbulence theory[7-9] developed in plasma physics, the spectrum cascade is shown to occur toward lower frequencies as a consequence of conservation of wave action. The cascade direction in wavenumber ($\mathbf{k}$) space is decided by the order of frequencies of interacting waves. Since many waves have dispersion properties such that the frequency $\omega$ increases with $|k|$, (i.e., $\partial\omega/\partial k>0$), the "inverse cascade" is a rather common phenomena in weak turbulence theory. However, it is the consequence of the cascade in $\omega$; hence, if $\partial\omega/\partial k<0$, the spectrum should cascade to a larger value of $|k|$. Even if there exists no general proof, weakly interacting waves have this property.

A drift wave[10] which exists in a nonuniform, magnetized plasma has interesting properties in this respect. It does not obey the standard wave equation since it does not have the symmetry property of a left and right going wave. The dispersion relation is given by $\omega\simeq T_e(\hat{\mathbf{z}}\times\mathbf{k})\cdot\nabla\ln n_0/eB_0$, where $\hat{\mathbf{z}}$ is the unit vector in the direction of the magnetic field, $n_0(\mathbf{x})$ is the plasma density, $T_e$ is the electron temperature, and $B_0$ is the ambient magnetic field. In a uniform plasma, $\omega=0$, similar to the hydrodynamic case. The wave exists only in the three-dimensional situation and its dispersion depends only on $\mathbf{k}\times\hat{\mathbf{z}}$; hence, it is expected to have a property distinct from the isotropic two-dimensional hydrodynamics.

In hydrodynamics, a situation similar to the drift wave exists when two-dimensional horizontal flow is coupled to gravity. There, three-dimensional dynamics are essential, even though the flow pattern can be described in terms of two-dimensional horizontal coordinates, because the vertical coordinate becomes a dependent variable.

In an inhomogeneous fluid, such a flow has a finite frequency similar to the drift wave. One well known example is the Rossby wave[11] in the atmospheric pressure system. There, the gradient in the Coriolis force works to produce a finite frequency.

Both the drift wave and the Rossby wave have properties which do not belong to either of these previously known cases in that they are pseudo-three-dimensional and have both strong (in large $k$ region) and weak (in small $k$ region) wave-wave interactions as will be shown. Hence, it is interesting to ask what is the nature of the turbulence produced by these waves. Does it obey an inverse cascade law as in two-dimensional hydrodynamic turbulence? Does it produce a cascade to shorter wavelengths as in the three-dimensional hydrodynamic case? Or a cascade in $\omega$ as for the weakly interacting waves?

In this paper we study this question. We first show in Secs. II and III that the drift wave turbulence and the Rossby wave turbulence obey identical nonlinear partial differential equations of the form

$$\frac{\partial}{\partial t}(\nabla^2\phi-\phi)-[(\nabla\phi\times\hat{\mathbf{z}})\cdot\nabla]\left(\nabla^2\phi+\ln\frac{\omega_c}{n_0}\right)=0\,. \qquad (1)$$

Here, $\omega_c$ and $\phi$ represent the ion cyclotron frequency and electrostatic potential for the drift wave and the Coriolis parameter and depth of the atmosphere for the Rossby wave. This equation was derived by Hasegawa and Mima[12] to describe drift wave turbulence; for convenience, we will subsequently refer to it as the Hasegawa–Mima equation.

In Sec. IV, we present conservation laws and the vortex solution of Eq. (1) and its dynamics and compare them with the two-dimensional Navier–Stokes vortices. In Sec. V, we present a decay rule among three interacting plane waves by considering the number of equivalent quanta. It is shown that the wavenumber splits into

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a larger and smaller value by the interaction, similar to two-dimensional Navier–Stokes turbulence.

In Sec. VI, we present the results of a numerical evaluation of the spectrum cascade using the decay rule for the three-wave interaction obtained in Sec. V. It is shown that the energy spectrum divides into two distinctive regions. One is the region of large wavenumbers, where the nonlinear term dominates the linear term. There, the spectrum is isotropic and the unidirectional energy spectrum obeys a $k^{-4}$ law as in the two-dimensional inertia range. The other region is the small wavenumber region where the linear term dominates. Here, the energy cascade occurs through resonant three-wave interactions which induce anisotropy in the spectrum.

The energy is found to condense at $k_x \simeq k_c$ and $k_y \simeq 0$, where $x$ is the direction of inhomogeneity (radial direction for a drift wave and latitudinal direction for a Rossby wave) and $k_c = (\kappa/4|\phi_0|)^{1/3}$; $\kappa$ is the measure of inhomogeneity and $\phi_0$ is the maximum amplitude of $\phi$. This result predicts the appearance of zonal flows in the $y$ (azimuthal) direction. The appearance of two regions in the Rossby wave spectrum was noted by Rhines[13] and the resulting anisotropy in the spectrum has also been studied by Holloway and Hendershott.[14] The appearance of an anisotropic spectrum has been noted by Williams[15] in his numerical analysis of nonlinear Rossby wave turbulence, a result clearly supported by the observed zonal flow pattern of Jupiter. The present result is the first verification of energy condensation to the zonal flow pattern through spectrum cascade.

## II. DRIFT WAVE

Consider an electrostatic wave at frequency $\omega$ much smaller than the ion cyclotron frequency $\omega_{ci}$ in a magnetized (with magnetic field $B_0\hat{z}$) and inhomogeneous [with density $n_0(\mathbf{x})$] plasma. A linear wave is known to exist in such a plasma if the phase velocity in the direction of the magnetic field, $\omega/k_z$, is between the electron and the ion thermal speed, $v_{Te}$ and $v_{Ti}$. The dispersion relation of the wave is given by $\omega = \mathbf{k}\cdot\mathbf{v}_d$, where $v_d$ is the diamagnetic drift velocity; the wave is called a drift wave.[10] It is sometimes called a universal mode because it is always excited by Cerenkov emission of electrons and is considered to play a crucial role in the magnetic confinement of a plasma.

The spectrum transfer in drift wave turbulence has been studied using nonlinear Landau damping of ions[16,17] or nonlinear mode coupling.[18] In either case, the nonlinearity which originates from the $\mathbf{E}\times\mathbf{B}$ drift is found to play the dominant role.

Here, we consider the mode coupling process. For the sake of comparison with the Rossby wave, it is convenient to assume that the ion temperature is much smaller than the electron temperature. For the drift wave description, it is convenient to use the small parameter $\epsilon$,

$$\epsilon = \frac{1}{\omega_{ci}}\frac{\partial}{\partial t} \sim \frac{1}{k_z v_{Te}}\frac{\partial}{\partial t} \sim \rho_s \left|\nabla\left(\ln\frac{n_0}{B_0}\right)\right| \sim \frac{|\Omega|}{\omega_{ci}}, \tag{2}$$

where $\Omega(=\nabla\times\mathbf{v})$ is the vorticity of the ion fluid,

$$\rho_s = (T_e/m_i)^{1/2}(\omega_{ci})^{-1}, \tag{3}$$

and $T_e$ is the electron temperature, $m_i$ is the ion mass, and $\mathbf{v}$ is the velocity field of ions.

The equations to describe the ion dynamics are the equation of motion for the cold ion fluid in an electrostatic field

$$\frac{d\mathbf{v}}{dt} = -\frac{e}{m_i}\nabla\phi + \mathbf{v}\times\boldsymbol{\omega}_{ci}, \tag{4}$$

and the equation of the number density conservation of ions

$$\nabla\cdot\mathbf{v} = -\frac{d}{dt}\ln n, \tag{5}$$

where $\mathbf{v}$ is the ion fluid velocity, $m_i$ is the ion mass, $e$ is the electric charge of the ion, $\phi$ is the electrostatic potential, and $\boldsymbol{\omega}_{ci}(\mathbf{x})$ $(=e\mathbf{B}_0(\mathbf{x})\hat{z}/m_i)$ is the vector ion cyclotron frequency.

The quasi-neutrality condition relates $n$ to the electron density which can be shown to obey the Boltzmann distribution

$$n = n_0(\mathbf{x})\exp(e\phi/T_e), \tag{6}$$

within the framework of the small parameter, Eq. (2).

From Eqs. (5) and (6) we have

$$\nabla\cdot\mathbf{v} = -\frac{d}{dt}\left(\ln n_0 + \frac{e\phi}{T_e}\right). \tag{7}$$

Since the drift wave is basically a vortex mode, we construct an equation for the vorticity $\Omega(=\nabla\times\mathbf{v})$ by taking the curl of Eq. (4). If we note

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}\nabla\mathbf{v}^2 - \mathbf{v}\times\boldsymbol{\Omega},$$

and

$$\nabla\times(\mathbf{v}\times\Omega) = -\Omega\nabla\cdot\mathbf{v} + (\Omega\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\Omega$$
$$= -\Omega\nabla_\perp\cdot\mathbf{v}_\perp - (\mathbf{v}\cdot\nabla)\Omega,$$

we have

$$\frac{d}{dt}(\Omega + \omega_{ci}) + (\Omega + \omega_{ci})\nabla_\perp\cdot\mathbf{v}_\perp = 0. \tag{8}$$

Here the subscript $\perp$ indicates the components perpendicular to the direction of the magnetic field, $\hat{z}$. Now, we assume a pseudo-three-dimensional situation such that

$$\left|\frac{\partial v_z}{\partial z}\right| = \epsilon|\nabla_\perp\cdot\mathbf{v}_\perp|. \tag{9}$$

This assumption is consistent with the condition of the existence of a drift wave. Physically, the assumption means that the ion inertia in the direction of the ambient magnetic field is negligible.

Equation (7) is then approximated by

$$\nabla_\perp\cdot\mathbf{v}_\perp = -\frac{d}{dt}\left(\ln n_0 + \frac{e\phi}{T_e}\right). \tag{7'}$$

If we substitute Eq. (7′) into Eq. (8) and use the small parameter expansion of Eq. (2), we have

$$\frac{d}{dt}\left(\ln\frac{\omega_{ci}}{n_0}+\frac{\Omega}{\omega_{ci}}-\frac{e\phi}{T_e}\right)=0\,. \tag{10}$$

Here, if we use the ordering of Eq. (2) the vorticity $\mathbf{\Omega}\cdot\hat{\mathbf{z}}$ is given by the $\mathbf{E}\times\mathbf{B}$ drift,

$$\Omega=(\nabla\times\mathbf{v}_\perp)\cdot\hat{\mathbf{z}}=\nabla\times\left(-\frac{\nabla\phi\times\hat{\mathbf{z}}}{B_0}\right)\cdot\hat{\mathbf{z}}=\frac{1}{B_0}\nabla^2\phi\,, \tag{11}$$

and

$$\frac{d}{dt}=\frac{\partial}{\partial t}-\frac{\nabla\phi\times\hat{\mathbf{z}}}{B_0}\cdot\nabla\,. \tag{12}$$

Equations (10), (11), and (12) form a closed set for the electrostatic potential $\phi$.

In a low $\beta$ plasma, the inhomogeneity in the magnetic field is regarded as small compared with that of the plasma density. If we take $\omega_{ci}$ to be approximately constant and use the following normalization for time, space, and $\phi$,

$$\omega_{ci}t\equiv t\,, \tag{13}$$

$$x/\rho_s\equiv x\,, \tag{14}$$

$$e\phi/T_e\equiv\phi\,, \tag{15}$$

Eqs. (10), (11), and (12) reduce to

$$\frac{\partial}{\partial t}(\nabla^2\phi-\phi)-[(\nabla\phi\times\hat{\mathbf{z}})\cdot\nabla]\left[\nabla^2\phi-\ln\left(\frac{n_0}{\omega_{ci}}\right)\right]=0\,, \tag{16}$$

which is the equation derived by Hasegawa and Mima.[12] Here, we suppressed the subscript $\perp$ in the gradient operator $\nabla$; $\nabla$ means

$$\nabla=\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}\,.$$

We note here that in a homogeneous case this expression, Eq. (16), closely resembles that for the stream function $\psi$ of the two-dimensional Euler equation for an incompressible fluid, which, in a homogeneous fluid, can be written

$$\frac{\partial}{\partial t}(\nabla^2\psi)-[(\nabla\psi\times\hat{\mathbf{z}})\cdot\nabla]\nabla^2\psi=0\,, \tag{17}$$

There exist two fundamental differences between these two equations. First, Euler's equation does not have a characteristic spatial scale, while the spatial scale of the Hasegawa–Mima equation is given by $\rho_s$ in Eq. (3). Second, the fluid motion of the Hasegawa–Mima equation is not divergence-free. There exists a sink or a source in the $x$, $y$ plane which is due to the implicit connection of the fluid in the $z$ direction.

In the presence of an inhomogeneity, the Hasegawa–Mima equation admits a linear wave whose dispersion relation is given by

$$\omega^*=-[(\mathbf{k}\times\hat{\mathbf{z}})\cdot\nabla\ln n_0]/(1+k^2)\,, \tag{18}$$

where $\mathbf{k}$ is the wave vector in the direction perpendicular to $\hat{\mathbf{z}}$. This is the well known drift wave frequency.

The typical size of the time and space parameters in a tokamak plasma are

$$\omega_{ci}\simeq10^8\sim10^9\ \text{sec}^{-1}\,,$$

$$\omega^*\simeq10^6\sim10^7\ \text{sec}^{-1}\,,$$

$$\rho_s\simeq10^{-3}\ \text{m}\,.$$

The size of the nonlinearities $e\phi/T_e$ and $\Omega/\omega_{ci}$ as measured by laser[19] and microwave scattering[20] from a tokamak plasma are

$$e\phi/T_e\simeq10^{-1}\sim10^{-2}\,,$$

$$\Omega/\omega_{ci}[=k^2\rho_s^2(e\phi/T_e)]\simeq10^{-2}\sim10^{-3}\,.$$

## III. ROSSBY WAVE

There exists a wave in the atmospheric pressure system which is almost identical in its properties to the drift wave. The wave is called a "Rossby wave," and it propagates by the gradient of the Coriolis force.

Let us consider the atmospheric motion on the surface of a rotating planet. The two-dimensional velocity of the atmospheric flow in the horizontal plane obeys the equation of motion,[21]

$$\frac{d\mathbf{v}}{dt}=-g\nabla h+f\mathbf{v}\times\hat{\mathbf{z}}\,, \tag{19}$$

where $f(\mathbf{x})$ is the Coriolis parameter, $h$ is the surface displacement of the atmosphere in the vertical ($\hat{z}$) direction, and $g$ is the constant of gravity. The quantity $h$ also represents the surface density of the atmosphere; hence, it obeys the continuity equation,

$$\nabla\cdot\mathbf{v}=-\frac{d}{dt}\ln H\,, \tag{20}$$

where $\nabla$ is again the two-dimensional differential operator in the $x$-$y$ plane and $H$ is the total depth of the atmosphere

$$H=H_0+h\,, \tag{21}$$

and $H_0$ is the average depth. We see immediately the close resemblance of Eqs. (19) and (20) to Eqs. (2) and (5). In fact, they are identical if $h\ll H_0$, and when $f$ is replaced by $\omega_{ci}$, and $h$ by $\phi$. The only difference is that the spatial variation of $f$ is generally larger than that of $H_0$. If we introduce a small parameter called the Rossby number,

$$\epsilon=\frac{1}{\langle f\rangle}\frac{\partial}{\partial t}\simeq\frac{\Omega}{\langle f\rangle}\simeq\left|\rho_g\nabla\ln\frac{H_0}{f}\right|\,, \tag{22}$$

and introduce the following scaling

$$\langle f\rangle t\equiv t\,, \tag{23}$$

$$x/\rho_g\equiv x\,, \tag{24}$$

$$h/H_0\equiv h\,, \tag{25}$$

where the spatial scale $\rho_g$ (Rossby radius) is given by

$$\rho_g=(gH_0)^{1/2}/\langle f\rangle\,, \tag{26}$$

and $\langle f\rangle$ is the average of $f$, Eqs. (19) to (21) can be reduced to the form of Eq. (1),

$$\frac{\partial}{\partial t}(\nabla^2h-h)-[(\nabla h\times\hat{\mathbf{z}})\cdot\nabla]\left(\nabla^2h-\ln\frac{H_0}{f}\right)=0\,. \tag{27}$$

The linearized solution gives the dispersion relation for the Rossby wave as

$$\omega_R = [(\mathbf{k} \times \hat{\mathbf{z}}) \cdot \nabla \ln f]/(1 + k^2) . \quad (28)$$

Stewart[22] and Morikawa[21] have studied the nonlinear dynamics of Rossby waves in terms on interacting vortices (geostrophic vortices) using Eqs. (19) and (20) in a uniform fluid. Typical parameters in the earth atmospheric system in the midlatitude are

$$H_0 \simeq 8 \times 10^3 \text{ m}, \quad \rho_g \simeq 2 \times 10^6 \text{ m},$$

$$\langle f \rangle \simeq 1.5 \times 10^{-4} \text{ sec}^{-1}, \quad \omega_R/f \simeq 10^{-1} .$$

Unlike the case of a magnetized plasma, where the finite ion gyroradius effects disallow the use of a simplified fluid expression for $k\rho_s \gg 1$ (if $T_i \simeq T_e$), in the case of geostrophic turbulence $k\rho_g$ can become very large.

Other than this point, we can see the close similarity between the drift wave and the Rossby wave.

## IV. CONSERVATION LAWS AND VORTEX SOLUTIONS

To consider the turbulence which can be described by the Hasegawa–Mima equation it is convenient to review some of the properties of this equation. First, it can easily be shown[12] that this equation contains two fundamental conserved quantities; the total energy,

$$\frac{\partial W}{\partial t} \equiv \frac{\partial}{\partial t} \int [(\nabla\phi)^2 + \phi^2] dV = 0 , \quad (29)$$

and the (potential) enstrophy,

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t} \int [(\nabla\phi)^2 + (\nabla^2\phi)^2] dV = 0 , \quad (30)$$

where $\int dV$ is the volume integral. Using these conserved quantities, a stationary spectrum $\langle |\phi_k|^2 \rangle$ for a loss-less case has been obtained[23]

$$\langle |\phi_k|^2 \rangle = (1 + k^2)^{-1} (\alpha + \beta k^2)^{-1} . \quad (31)$$

Hence, the energy spectrum becomes

$$W_k = (1 + k^2) |\phi_k|^2 = (\alpha + \beta k^2)^{-1} , \quad (32)$$

and is the same as for the case of the two-dimensional Euler equation.[1] If $\alpha\beta < 0$, the spectrum indicates a negative temperature state and condensation at a small $k$ value ($= |\alpha/\beta|$).

Assuming a certain rigidity of the vortices, Stewart[22] and Morikawa[21] studied interactions among vortex solutions of Eq. (1) without the inhomogeneous term. The formulation is similar to the two-dimensional vortex solutions.[1] The solution can be written

$$\phi = \sum_j \omega_j K_0(|\mathbf{r} - \mathbf{r}_j(t)|) , \quad (33)$$

and

$$\frac{d\mathbf{r}_i}{dt} = -\nabla\phi \times \hat{\mathbf{z}} , \quad (34)$$

where $K_0$ is the modified Bessel function of the second kind, and $\omega_j$ is the circulation.

In contrast, the two-dimensional Euler's vortex solution is $\omega_j \ln|\mathbf{r} - \mathbf{r}_j|$. If $|\mathbf{r} - \mathbf{r}_i| \ll 1$, these two solutions are the same, but Stewart's vortex dies off exponentially at $|\mathbf{r} - \mathbf{r}_i| \geq 1$, hence, it can be considered as a shielded vortex. This means that the vortex of Eq. (1) has a finite size given by $\rho_s$ (or $\rho_g$). This has a significant effect on the convection.

The interacting $n$ vortices of the homogeneous Hasegawa–Mima equation can be described by the Hamiltonian,

$$H = \sum_{i>j} \sum \omega_i \omega_j K_0(|\mathbf{r}_i - \mathbf{r}_j|) ; \quad (35)$$

here, the canonical variables are the $x$ and $y$ coordinates of the centers of the vortices, $x_i$, $y_i$. The Hamiltonian equations of motion for these centers are

$$\frac{\partial H}{\partial x_i} = \omega_i \dot{y}_i , \quad \frac{\partial H}{\partial y_i} = -\omega_i \dot{x}_i . \quad (36)$$

It can easily be verified that the following conservation relations exist;

(i) invariance of $H$:

$$\frac{dH}{dt} = 0 ,$$

(ii) stationary mass center:

$$\frac{d}{dt} \sum_i \omega_i \mathbf{r}_i = 0 ,$$

(iii) conservation of total angular momentum:

$$\frac{d}{dt} \sum_i \omega_i (\mathbf{r}_i \times \mathbf{v}_i) = 0 ,$$

(iv) conservation of moment of inertia:

$$\frac{d}{dt} \sum_i \omega_i r_i^2 = 0 .$$

We note here that these solutions of interacting vortices break down when the vortex density is increased and the inelastic collision starts to dominate. As the density is further increased, the field becomes turbulent.

## V. SPECTRUM CASCADE

Here, we consider the turbulence property of the Hasegawa–Mima equation. It is convenient to consider the dynamic change of the spatial Fourier spectrum. If we write

$$\phi(\mathbf{x}, t) = \frac{1}{2} \sum_k [\phi_\mathbf{k}(t) \exp(i\mathbf{k} \cdot \mathbf{x}) + \text{c.c.}] , \quad (37)$$

Eq. (1) becomes

$$\frac{d\phi_\mathbf{k}}{dt} + i\omega_\mathbf{k}\phi_\mathbf{k} = \sum_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''=0} \Lambda^\mathbf{k}_{\mathbf{k}',\mathbf{k}''} \phi^*_{\mathbf{k}'} \phi^*_{\mathbf{k}''} , \quad (38)$$

where the asterisk indicates the complex conjugate,

$$\omega_\mathbf{k} = -\frac{\mathbf{k} \times \hat{\mathbf{z}} \cdot \nabla \ln(n_0/\omega_{ci})}{1 + k^2} , \quad (39)$$

is the drift or Rossby wave frequency and the matrix element $\Lambda^\mathbf{k}_{\mathbf{k}',\mathbf{k}''}$ is given by

$$\Lambda^\mathbf{k}_{\mathbf{k}',\mathbf{k}''} = \frac{1}{2} \frac{1}{1 + k^2} (\mathbf{k}' \times \mathbf{k}'') \cdot \hat{\mathbf{z}} (k''^2 - k'^2) . \quad (40)$$

Let us consider three plane waves with wavenumbers

$\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ such that $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=0$. Let us suppose that these waves have amplitudes larger than other waves in the summation of Eq. (38) and study the energy flow among these three waves.[24] Equation (38) for the three interacting waves may be written

$$\frac{d\phi_1}{dt}+i\omega_1\phi_1=\Lambda^1_{2,3}\phi_2^*\phi_3^* , \tag{41}$$

$$\frac{d\phi_2}{dt}+i\omega_2\phi_2=\Lambda^2_{3,1}\phi_3^*\phi_1^* , \tag{42}$$

$$\frac{d\phi_3}{dt}+i\omega_3\phi_3=\Lambda^3_{1,2}\phi_1^*\phi_2^* , \tag{43}$$

where

$$\phi_j(t)=\phi_{\mathbf{k}_j}(t) , \tag{44}$$

and

$$\omega_j=\omega_{\mathbf{k}_j}, \quad j=1,2,3 . \tag{45}$$

The direction of energy flow or decay may be found by studying the stability of a situation in which one of the modes, 1, 2, 3 is more highly populated than others. For this purpose we first assume, without loss of generality, that $k_j\equiv|\mathbf{k}_j|$ such that

$$k_1\le k_2\le k_3 . \tag{46}$$

We first consider a case in which the $\mathbf{k}_2$ mode is highly populated so that $|\phi_2|\gg|\phi_1|$, $|\phi_3|$. We can then linearize Eqs. (41)–(43) to give,

$$\phi_2=A_2\exp(-i\omega_2 t), \quad A_2=\text{const} , \tag{42$'$}$$

and

$$\frac{dA_1}{dt}=\Lambda^1_{2,3}A_2^*A_3^*\exp(i\theta t) , \tag{41$'$}$$

$$\frac{dA_3}{dt}=\Lambda^3_{1,2}A_2^*A_1^*\exp(i\theta t) , \tag{43$'$}$$

with

$$\phi_j\equiv A_j(t)\exp(-i\omega_j t), \quad j=1,3$$

and

$$\theta=\omega_1+\omega_2+\omega_3 , \tag{47}$$

is the frequency mismatch.

From Eqs. (41′) and (43′), we have

$$\frac{d^2A_1}{dt^2}-i\theta\frac{dA_1}{dt}-\Lambda^1_{2,3}\Lambda^3_{1,2}|A_2|^2A_1=0 . \tag{48}$$

Hence, the instability (exponential growth of $A_1$ and $A_3$) occurs when

$$\theta^2-4\Lambda^1_{2,3}\Lambda^3_{1,2}|A_2|^2<0; \tag{49}$$

and the decay rate $\gamma$ is given by

$$\gamma=(\Lambda^1_{2,3}\Lambda^3_{1,2}|A_2|^2-\tfrac14\theta^2)^{1/2} . \tag{50}$$

Inequality (49) shows that the stability is decided by the sign of the product $\Lambda^1_{2,3}\Lambda^3_{1,2}$.

Now, in view of the assumed relation (46), both of the quantities $k_2^2-k_3^2$ and $k_1^2-k_2^2$ are negative (or zero) in Eq. (40), and $(\mathbf{k}_2\times\mathbf{k}_3)\cdot\hat{\mathbf{z}}$ and $(\mathbf{k}_1\times\mathbf{k}_2)\cdot\hat{\mathbf{z}}$ have the same sign (if not zero). Hence, $\Lambda^1_{2,3}\Lambda^3_{1,2}>0$ and this situation can be unstable.

On the other hand, since $\Lambda^2_{3,1}\Lambda^3_{1,2}$ and $\Lambda^1_{2,3}\Lambda^2_{3,1}$ are always negative (or zero), if modes 1 or 3 are highly populated, the system is stable. Hence, we conclude that the necessary condition for a spectrum cascade is to excite a shorter and a lower wavelength mode simultaneously. We note here that since this is not a resonant decay, by the time the decay process is completed, many other modes have also been excited.

Let us discuss the conservation of quanta in the decay process. If we introduce a number of quanta of the three waves, defined by

$$N_p=(1+k_p^2)|\phi_p|^2/|k_q^2-k_r^2| , \quad k_q^2\ne k_r^2 , \tag{51}$$

from Eqs. (41) to (43), we find

$$N_3-N_1=\text{const} ,$$

and

$$N_2+N_3=\text{const}, \quad N_1+N_2=\text{const} . \tag{52}$$

These relations show that a loss of one quantum in $N_2$ appears as a gain of one quantum in $N_1$ and $N_3$, respectively. The quantity $N$ defined in Eq. (51) thus serves as the role of number of quanta in the decay process. Since the characteristic frequency of a vortex is zero, the standard definition of a number of quanta, energy/$\hbar\omega$, does not apply here. The number of quanta defined here has the strange property that it depends on the wavenumbers of the other interacting waves.

In a region of small wavenumbers, the first term in the decay rate expression, Eq. (50), becomes small and decay occurs only when $\theta\simeq 0$, i.e., when the frequency mismatch is small. In this case the resonant three-wave interaction dominates the decay process. The number of quanta defined in Eq. (51) reduces to the conventional form because

$$k_q^2-k_r^2=k_{qy}/\omega_q-k_{ry}/\omega_r=\omega_p M , \tag{53}$$

where

$$M=(3\omega_p\omega_q\omega_r)^{-1}[\omega_p(k_{ry}-k_{qy})+\omega_q(k_{py}-k_{ry}) + \omega_r(k_{qy}-k_{py})]$$

and $k_y$ is the component of the $\mathbf{k}$ vector in the direction of $\hat{\mathbf{z}}\times\nabla\ln n_0$. Equation (53) with Eq. (51) gives $N_p \propto W_p/\hbar\omega_p$. Hence, in the long wavelength region, decay occurs from a highest frequency mode to two lower frequency modes.

Now, the energy $W_\mathbf{k}$ of the $\mathbf{k}$ mode is given by $W_k=|\phi_\mathbf{k}|^2(1+k^2)$. Hence, from Eqs. (51) and (52), we see that the partition of energy of modes 1 and 3, for the loss of an energy $\Delta W_2=1$ of mode 2 is given by

$$\Delta W_1=\frac{k_3^2-k_2^2}{k_3^2-k_1^2}, \quad \Delta W_3=\frac{k_2^2-k_1^2}{k_3^2-k_1^2} . \tag{54}$$

In summary, the cascade occurs from the wave with wavenumber $k_2$ such that $k_1<k_2<k_3$ to waves with wavenumbers $k_1$ and $k_3$. If the frequency mismatch, $\theta(=\omega_1+\omega_2+\omega_3)$, is zero the cascade occurs from the wave with the highest frequency $\omega_2(=-\omega_1-\omega_3)$ to waves with lower frequencies $\omega_1$ and $\omega_3$.

## VI. SPECTRUM DISTRIBUTION

In this section we present a computer experiment of spectrum cascade and show the resulting stationary spectra. In Sec. V, it was shown that a three-wave decay occurs when $\gamma$ in Eq. (50) is real. A necessary condition for decay is that the wavenumber of the decaying wave must have a magnitude which is between those of the wavenumbers of the waves to which it decays. The computer experiment utilizes this decay concept. A wave with amplitude larger than that of the stationary spectrum at the corresponding wavenumber will decay to two other waves, if the wavenumbers of these waves satisfy the decay condition.

We designed the computer experiment in the following way. First, we assumed a wave at $k_y=1$ and $k_x=0$ with energy $W=1$. We then chose a random triangle of $\mathbf{k}$ vectors with one side corresponding to $k_y=1$, $k_x=0$. If the other two sides (call them $\mathbf{k}=\mathbf{k}_1$ and $\mathbf{k}=\mathbf{k}_3$), had values of $\Lambda$ and $\theta$ that would make $\gamma$ real in Eq. (50), we allow the decay. We calculate energy at wave vectors $\mathbf{k}_1$ and $\mathbf{k}_3$ using the partition formula, Eq. (54). The only parameter needed here is $\kappa(=\nabla \ln n_0)$, which we take to be constant, assuming an exponential density gradient. If $\gamma$ happened to be imaginary, we choose another set of $\mathbf{k}_1$ and $\mathbf{k}_3$ until $\gamma$ becomes real. After the initial wave decays to two waves, we repeat the procedure, i.e., we choose a random triangle with one side now given by $\mathbf{k}_1$ (or $\mathbf{k}_3$) and check the decay condition by evaluating the value of $\gamma$ now using the new amplitude of the $\mathbf{k}_1$ (or $\mathbf{k}_3$) mode obtained from the energy partition law in the previous step. After several cascades, the energy spectrum spreads out. To obtain the inertia range spectrum at $k>1$, we set a maximum value of $k(=k_m)$ and when a mode with $k>k_m$ is born, it is taken away and its energy is reintroduced at $k_y=1$ so that the total energy is conserved, but with loss of the potential enstrophy. This technique should produce an inertia range spectrum for the enstrophy. Normally, we chose $k_m=10^2$. As the energy cascades to smaller values of $k$, the matrix element $\Lambda$ decreases rapidly and the decay condition becomes more difficult to satisfy. The decay requires the condition of frequency matching such that $\theta\simeq 0$ in Eq. (50). Since $\theta$ depends only on $k_y$ [cf. Eqs. (18) and (28)], the anisotropy of the spectrum evolves at a smaller value of $k$. With respect to the $k_y$ dependency of the spectrum, since the energy tends to decay to lower frequencies, hence to smaller $k_y$, the spectrum tends to condense at $k_y\simeq 0$. On the other hand, the cascade in the $k_x$ plane tends to stop at a critical value of $k_x(=k_c)$ because a further cascade into smaller values of $k_x$ does not lower the frequency, and in addition, reduces the coupling coefficient. Thus, the energy spectrum tends to condense at $k_x\simeq k_c$, where $k_c$ is roughly obtained by $\gamma=0$ for an isotropic value of $k$, i.e.,

$$\frac{k_c^8}{(1+k_c^2)^2}\left|\phi_0\right|^2\simeq\frac{\kappa^2}{4}\frac{k_c^2}{(1+k_c^2)^2},$$

or

$$k_c=(\kappa/4|\phi_0|)^{1/3}, \tag{55}$$

where $\phi_0$ is the amplitude of the initial wave ($\phi_0=1/\sqrt{2}$ for the choice of initial $W=1$).

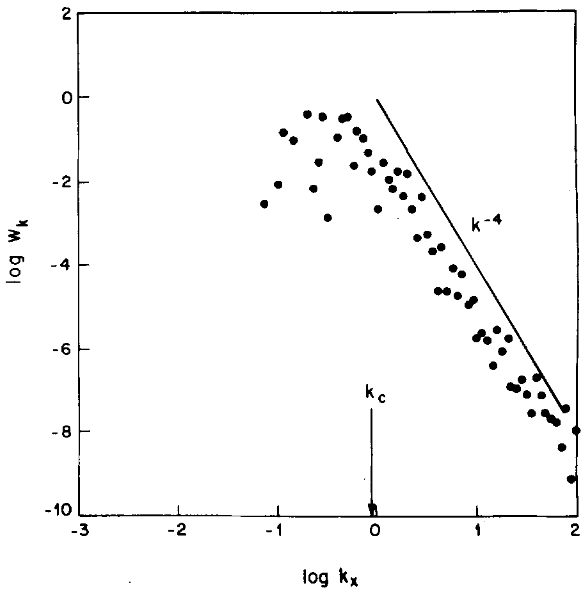


FIG. 1. Unidirectional energy spectrum $W_k=(1+k^2)|\phi_0|^2$ obtained using the cascade model plotted as a function of $k_x$, where $x$ is the direction of the inhomogeneity for the case with $k_c=0.89$ [$k_c$ is defined in Eq. (55)]. Note that the spectrum obeys $k^{-4}$ at $k\gg k_c$, and has a peak at $k\simeq k_c$.

The only parameter that controls the spectrum is $k_c$. The quantity $k_c^{-1}$ play a role somewhat similar to the Reynold's number since if $k_c^{-1}\to\infty$, the system is inherently turbulent, while if $k_c^{-1}\to 0$, the linear drift waves dominate the dynamics.

The energy spectra obtained using the computer experiment described here are shown in Figs. 1–4. Figures 1 and 2 are the results for $k_c=0.89$. Figure 1 is the energy spectrum obtained at the 59th step of cascade plotted as a function of $k_x$, while Fig. 2 is that plotted for $k_y$. We see that at $k\gg k_c$, the spectrum is isotropic and both $k_x$ and $k_y$ obey Kraichnan's inertial range

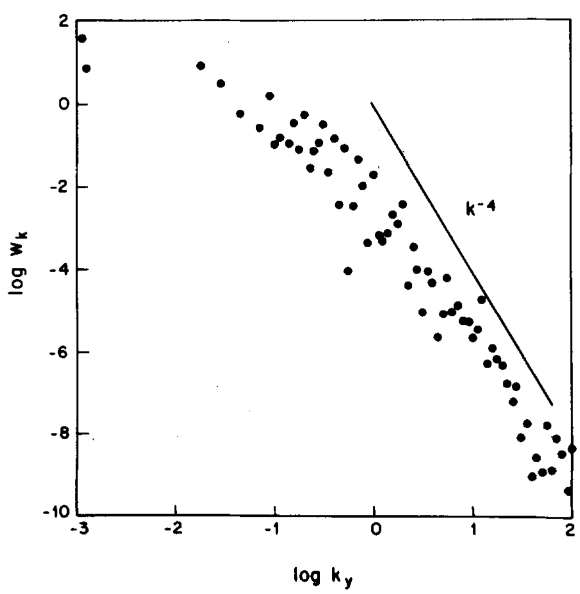


FIG. 2. $W_k$ for $k_c=0.89$ plotted as a function of $k_y$. The spectrum at $k\gg k_c$ is isotropic and obeys $k^{-4}$. The energy cascades to $k_y\to 0$.

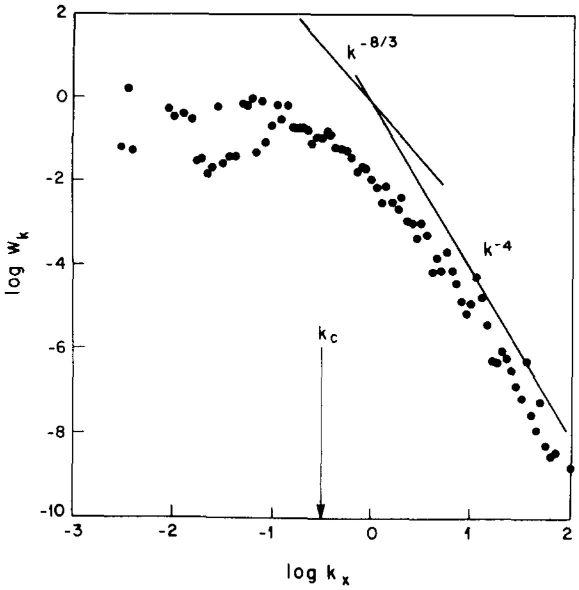


FIG. 3. $W_k$ vs $k_x$ for $k_c = 0.28$, indicating a higher level of turbulence than in Figs. 1 and 2. The hydrodynamic turbulence region is extended to a smaller value of $k$.

spectrum $W_k \sim k^{-4}$. This result is in agreement with the numerical solution of the Hasegawa–Mima equation for a uniform case obtained by Fyfe and Montgomery[25] and with the results of Prater *et al.*[26] and Basdevant and Sadourny[27] for the Navier–Stokes equation. This result justifies the applicability of the present rather empirical method.

As is expected at $k \lesssim k_c$, the spectrum develops anisotropy. The energy continuously cascades to smaller values of $k_y$ and tends to condense as $k_y \to 0$, while it tends to accumulate at $k_x \simeq k_c$. The spectrum obtained here has a qualitative agreement with the observation in ATS tokamak plasmas.[19]

When $k_c$ is decreased, the hydrodynamic turbulent region expands into smaller values of $k$. An example of the energy spectrum for such a case is shown in Figs. 3 and 4. Figures 3 and 4 are the energy spectrum obtained at the 80th step of the decay plotted versus $k_x$ and $k_y$ for $k_c = 0.28$. In this case $1 > k > k_c$, the gradient becomes weaker which may indicate the appearance of the inertia range of energy cascade,[25] i.e., $W_k \simeq k^{-8/3}$.

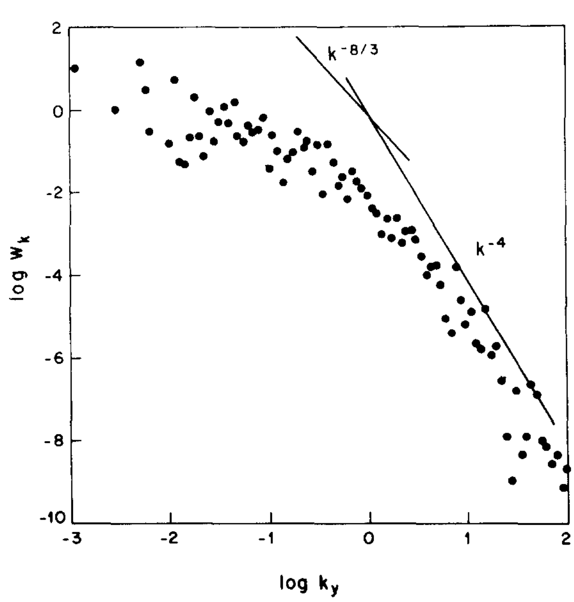


FIG. 4. $W_k$ vs $k_y$ for $k_c = 0.28$.

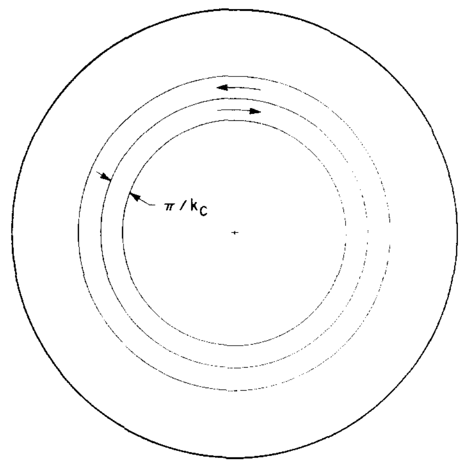


FIG. 5. Predicted zonal flow pattern in a magnetized cylindrical plasma which appears as a consequence of energy cascade and condensation at $k_y = 0$ and $k_x \simeq k_c$.

The fact that the stationary energy spectrum peaks at $k_x = k_c$ and $k_y = 0$, indicates that the energy condensates to zonal flows which are periodic in $x$, the direction of the density gradient for a drift wave and of the Coriolis parameter for a loss by wave. The appearance of zonal flows was noted by Williams[15] who numerically solved the nonlinear Rossby wave equation.

In a magnetized cylindrical plasma, the zonal flows appear in the azimuthal direction with a radially periodic structure with periodicity $2\pi/k_c$ as shown in Fig. 5. The present result has a significant implication for plasma confinement by magnetic fields. Drift wave turbulence which always seems to be present in a nonuniform plasma has been predicted to produce anomalous diffusion of plasma particles. However, if zonal flow is the ultimate form of drift wave turbulence, it will inhibit particle transport across the flow and the plasma will be well confined, even in a highly turbulent state. One evidence of such a state is the zonal flows of the planet Jupiter.[15] The clear longitudinal patterns of different color zones are a strong indication of little transport across the zonal flows.

Recent experimental results seem to indicate that particle confinement is not correlated with a level of turbulence.[26] These results may be explained by the appearance of zonal flows in drift wave turbulence which inhibit radial transport.

## VII. CONCLUSION AND DISCUSSION

We have discussed a variety of subjects in this paper, hence the conclusion cannot be stated in a simple paragraph. We therefore list the following important conclusions that we have obtained in this work.

(1) We have shown that both the drift wave turbulence and the Rossby wave turbulence can be described by a nonlinear partial differential equation with the structure

of Eq. (1). Hence, aside from the physical scaling in time and space, these two waves should have similar properties both in the linear and nonlinear regions.

(2) For a uniform case, the model equation admits a solution with interacting vortices. This aspect has been well known in geostrophic vortices in the atmospheric pressure system, however, it is less well known in the case of the drift wave in a magnetized plasma. This analog indicates that the vortex dynamics in drift wave turbulence are also important.

(3) We have derived a new cascading model based on the decay of a plane wave to other plane waves of the Hasegawa–Mima model equation. By obtaining $k$ values of two waves produced by the decay one can trace the cascade process. These $k$ values also decide the partition of energy and enstrophy of the newly created waves. The cascade process therefore gives the energy and enstrophy transfer in $k$ space. Using this new technique, energy spectra on the drift wave and Rossby wave turbulence are obtained.

(4) The cascade model has produced an energy spectrum which has two distinctive regions. One is at $k \gg k_c$ [$k_c$ defined in Eq. (55)] where the unidirectional spectrum $W_k$ is isotropic, and is given by $k^{-4}$; this region corresponds to the inertia range spectrum of two-dimensional hydrodynamic turbulence. The other region is for $k < k_c$ where the spectrum is anisotropic and obeys the mode of the resonant three wave interaction.

(5) The spectrum tends to condense at $k_y \simeq 0$ and $k_x \simeq k_c$ which predicts the formation of zonal flows in the $y$ direction which are periodic in the $x$ direction. The formation of such zonal flows has a potential implication on the effect of drift wave turbulence on particle transport since the zonal flows may inhibit particle transport across the flow.

## ACKNOWLEDGMENTS

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