# Plasma Instabilities

## Solutions for Exercises Series 6

*Ballooning and interchange modes*

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1. Use the general definition for the grad operator,

$$\nabla\beta = \nabla r\frac{\partial\beta}{\partial r} + \nabla\omega\frac{\partial\beta}{\partial\omega} + \nabla\phi\frac{\partial\beta}{\partial\phi}$$

Setting,

$$\nabla\beta = \nabla\phi + a\nabla r + b\nabla\omega$$

with

$$\beta \equiv \phi - q(\psi)\,\theta(\omega) \quad \text{where} \quad \theta(\omega) = \omega - (\epsilon + \Delta')\sin\omega. \quad \text{Thus:}$$

we have

$$a = \frac{\partial\beta}{\partial r} = q\sin\omega\frac{1}{r}(\epsilon + r\Delta'') - \frac{q}{r}s[\omega - (\epsilon+\Delta')\sin\omega] \quad \text{and} \quad b = \frac{\partial\beta}{\partial\omega} = -q[1-(\epsilon+\Delta')\cos\omega],$$

with $s = (r/q)dq/dr$. A small amount of algebra gives

$$(\nabla\beta)^2 = (\nabla\phi)^2 + a^2(\nabla r)^2 + b^2(\nabla\omega)^2 + 2ab\nabla r\cdot\nabla\omega$$

and also,

$$(\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\omega) = \psi'\nabla r\cdot\nabla\theta\left[a^2(\nabla r)^2 + b^2(\nabla\omega)^2\right] + \psi' ab\left[(\nabla\omega)^2(\nabla r)^2 + (\nabla r\cdot\nabla\theta)^2\right].$$

Hence we easily obtain

$$(\nabla\beta)^2\nabla\psi\cdot\nabla\omega - (\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\omega) = \left[(\nabla\phi)^2\nabla r\cdot\nabla\omega + ab(\nabla r\cdot\nabla\omega)^2 - ab(\nabla r)^2(\nabla\omega)^2\right]\psi',$$

Note that $(\nabla\phi)^2 = 1/R^2$ so this term is smaller than that of $ab(\nabla r)^2(\nabla\omega)^2$ by factor $\epsilon^2$. Use also that $\nabla\psi\cdot\nabla\omega = \psi'\nabla r\cdot\nabla\omega \sim \psi'(1/r)\epsilon\sin\omega$ (see lecture notes and exercises for week 2). So $(\nabla\psi\cdot\nabla\omega)^2$ is also smaller than that of $ab(\nabla r)^2(\nabla\omega)^2$.

Lifting results from exercise week 2, we have $(\nabla r)^2(\nabla\omega)^2 = (1/r^2)(1+O(\epsilon\cos\omega))$, we thus obtain,

$$\begin{aligned}(\nabla\beta)^2\nabla\psi\cdot\nabla\omega - (\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\omega) &\approx \psi' ab(\nabla r)^2(\nabla\omega)^2 \\ &= \frac{q^2(\psi')}{r^3}\left[(\epsilon + r\Delta'')\sin\omega - s\omega + O(\epsilon,\epsilon\cos\theta)\right]\end{aligned}$$

2. Simply use the definitions in the question, and notice that the $P'$ term on the curvature cancels with that of $P'$ in $F_2'$, and the shear in $F_2'$ cancels with shear in the poloidal component of $(B^2)'$. We obtain,

$$\begin{aligned}\frac{1}{B^2}\frac{\partial}{\partial r}\left(\frac{B^2}{2}+P\right) &= \frac{R^2}{2}\frac{\partial}{\partial r}\left(\frac{1}{R^2}\right) + \frac{1}{2}\frac{d}{dr}\left(2F_2(r) + \frac{\epsilon^2}{q^2}(1+2\Delta'\cos\omega) + \frac{2P}{B_0^2}\right) + O(\epsilon^2/R_0) \\ &-\frac{1}{2R^2}\frac{\partial R^2}{\partial r} + \frac{1}{2}\frac{d}{dr}\left(2F_2(r) + \frac{\epsilon^2}{q^2}(1+2\Delta'\cos\omega) + \frac{2P}{B_0^2}\right) + O(\epsilon^2/R_0) \\ &= \frac{1}{R_0}\left[-\cos\omega + \epsilon\cos^2\omega - \frac{\epsilon}{q^2} + O(\Delta', r\Delta''\epsilon\cos\omega,\epsilon^2)\right]\end{aligned}$$

where we have used, as noted above,

$$\frac{dF_2}{dr} = -\frac{1}{B_0^2}\frac{dP}{dr} - \frac{r}{R_0^2 q^2}(2-s).$$

Also,

$$\frac{1}{B^2}\frac{\partial}{\partial\omega}\left(\frac{B^2}{2}+P\right) = \left(\epsilon\sin\omega + O(\epsilon^2)\right)$$

Hence, we obtain, on noting $\epsilon(\sin^2\omega + \cos^2\omega) = \epsilon$ (which gives the crucial toroidal Mercier contribution):

$$\kappa_w = -\frac{1}{\psi' R_0}\left[\cos\omega - \epsilon\left\{1-\frac{1}{q^2}\right\} + \sin\omega(s\omega - r\Delta''\sin\omega) + O(\epsilon\sin\omega, \epsilon\cos\omega)\right].$$

3. Use that

$$(\boldsymbol{\nabla}\beta)^2 = (\boldsymbol{\nabla}\phi)^2 + a^2(\boldsymbol{\nabla}r)^2 + b^2(\boldsymbol{\nabla}\omega)^2 + 2ab\boldsymbol{\nabla}r\cdot\boldsymbol{\nabla}\omega$$

and substitute for $a$, $b$ etc to easily give

$$(\boldsymbol{\nabla}\beta)^2 = \frac{q^2}{r^2}\left[1 + (s\omega - r\Delta''\sin\omega)^2 + O(\epsilon\sin\omega, \epsilon\cos\omega)\right]$$

To continue for the calculation of

$$\frac{1}{\mathcal{J}_{\psi,\omega}}\frac{\partial}{\partial\omega}\left[\left(\frac{\boldsymbol{\nabla}\beta}{B}\right)^2\frac{1}{\mathcal{J}_{\psi,\omega}}\frac{\partial}{\partial\omega}X\right] + 2\kappa_w\frac{dP}{d\psi}X = 0$$

it is sufficient to use $\mathcal{J}_{\psi,\omega} = (\psi')^{-1}R_0 r$. Together with the results from this question, and the previous ones, and using $r\Delta'' \to \alpha$ we easily obtain

$$\frac{\partial}{\partial\omega}\left[\left\{1+(s\omega-\alpha\sin\omega)^2 + O(\epsilon\sin\omega,\epsilon\cos\omega)\right\}\frac{\partial}{\partial\omega}X\right] + \alpha\left[\cos\omega - \epsilon\left\{1-\frac{1}{q^2}\right\} + \sin\omega(s\omega-\alpha\sin\omega) + O(\epsilon\sin\omega,\epsilon\cos\omega)\right]X = 0.$$

Note that non-orthogonal corrections not included would appear in the $O$ sinusoidal terms.

4. The reduced ballooning equation is obtained by neglecting the epsilon corrections:

$$\frac{\partial}{\partial\omega}\left[\left\{1+(s\omega-\alpha\sin\omega)^2\right\}\frac{\partial}{\partial\omega}X\right] + \alpha\left[\cos\omega + \sin\omega(s\omega-\alpha\sin\omega)\right]X = 0.$$

5. Writing

$$\frac{\partial}{\partial\omega}\left[\left\{1+(s\omega-\alpha\sin\omega)^2 + O(\epsilon\sin\omega,\epsilon\cos\omega)\right\}\frac{\partial}{\partial\omega}X\right] + \alpha\left[\cos\omega - \epsilon\left\{1-\frac{1}{q^2}\right\} + \sin\omega(s\omega-\alpha\sin\omega) + O(\epsilon\sin\omega,\epsilon\cos\omega)\right]X = 0$$

in the form

$$\frac{d}{d\omega}\left[f\frac{dX}{d\omega}\right] + gX = 0,$$

with

$$f = a + b\omega + c\omega^2 \quad \text{and} \quad g = d + e\omega,$$

and $a$, $b$, $c$, $d$ and $e$ are $2\pi$ are periodic functions of $\omega$, we identify by inspection,

$$\begin{aligned}
a &= 1 + \alpha^2\sin^2\omega + O(\epsilon\sin\omega,\epsilon\cos\omega)\\
b &= -2\alpha s\sin\omega\\
c &= s^2\\
d &= \alpha\{\cos\omega - \epsilon[1-1/q^2)] - \alpha\sin^2\omega\} + O(\epsilon\sin\omega,\epsilon\cos\omega)\\
e &= \alpha s\,\sin\omega.
\end{aligned}$$

6. Define

$$X = \omega^p \left[ X_0(\omega) + \frac{X_1(\omega)}{\omega} + \frac{X_2(\omega)}{\omega^2} \right],$$

expand out $X'$ and $X''$, and use $f = a + b\omega + c\omega^2$ and $g = d + e\omega$ in

$$\frac{d}{d\omega}\left[ f\frac{dX}{d\omega}\right] + gX = 0 \tag{1}$$

First, using these expansions, identify the $\omega^{p+2}$ coefficient of Eq. (1) to obtain,

$$c'X_0' + cX_0'' = 0.$$

That is $(cX_0')' = 0$. Integrating twice we have,

$$X_0 = C_1 \int \frac{d\omega}{c} + C$$

where $C_1$ and $C$ are constants. Since $X_0$ is periodic, and $c$ is periodic, we must have $C_1 = 0$. Hence $X_0 = C$ a constant. Henceforth we set the constant $C = 1$, so that $X_0 = 1$.
Next, identify the $\omega^{p+1}$ coefficient of Eq. (1), using $X_0 = 1$. We obtain,

$$[c(X_1') + p]' + e = 0.$$

Integrating we have that,

$$cX_1' = -(cp + \hat{e}) + C_3$$

with $C_3$ a constant, and $\hat{e}$ is defined in the question. To find this constant, divide the last equation by $c$ and integrate over $2\pi$. Use that $\langle X_1' \rangle = 0$ to give,

$$C_3 = \frac{p + \langle \hat{e}/c \rangle}{\langle 1/c \rangle}$$

so that

$$\frac{dX_1}{d\omega} = X_1' = -\left(p + \frac{\hat{e}}{c}\right) + \frac{p + \langle \hat{e}/c \rangle}{c\,\langle 1/c \rangle}.$$

Next, identify the $\omega^p$ coefficient of Eq. (1). One obtains

$$(p+1)c(X_1' + p) + \{c[X_2' + (p-1)X_1] + b(X_1' + p)\}' + eX_1 + d = 0.$$

Averaging over $\omega$ eliminates the $c[X_2' + (p-1)X_1] + b(X_1' + p)$ contribution, thus eliminating $X_2$ entirely, giving

$$0 = \langle (p+1)c(X_1' + p) + eX_1 + d \rangle$$

On substituting the result for the $\omega^{p+1}$ coefficient (substituting $X_1'$) we have:

$$(p+1)\left\{ -\left(p\langle c \rangle + \langle \hat{e} \rangle\right) + \frac{p + \langle \hat{e}/c \rangle}{\langle 1/c \rangle} \right\} + p(p+1)\langle c \rangle + \langle eX_1 \rangle + \langle d \rangle = 0.$$

7. Obtain $\langle \hat{e}X_1' \rangle$ from

$$X_1' = -\left(p + \frac{\hat{e}}{c}\right) + \frac{p + \langle \hat{e}/c \rangle}{c\,\langle 1/c \rangle},$$

i.e.

$$\langle \hat{e}X_1' \rangle = -\left(p\langle \hat{e} \rangle + \left\langle \frac{\hat{e}\hat{e}}{c} \right\rangle\right) + \langle \hat{e}/c \rangle \left( \frac{p + \langle \hat{e}/c \rangle}{\langle 1/c \rangle} \right).$$

Use also the result from the previous question,

$$-\langle eX_1\rangle = (p+1)\left\{-\langle \hat{e}\rangle + \frac{p+\langle \hat{e}/c\rangle}{\langle 1/c\rangle}\right\} + \langle d\rangle$$

and $\langle \hat{e}X_1'\rangle = -\langle eX_1\rangle$ to yield a quadratic equation in $p$:

$$p^2+p+D_M=0, \quad D_M = \left\langle \frac{\hat{e}}{c}\right\rangle - \left\langle \frac{\hat{e}}{c}\right\rangle^2 + \left\langle \frac{1}{c}\right\rangle \left(\left\langle \frac{\hat{e}\hat{e}}{c}\right\rangle + \langle d\rangle - \langle \hat{e}\rangle\right).$$

This of course yields,

$$p = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - D_M}, \quad \text{with, } D_M = \left\langle \frac{\hat{e}}{c}\right\rangle - \left\langle \frac{\hat{e}}{c}\right\rangle^2 + \left\langle \frac{1}{c}\right\rangle \left(\left\langle \frac{\hat{e}\hat{e}}{c}\right\rangle + \langle d\rangle - \langle \hat{e}\rangle\right).$$

8. If $D_M < 1/4$ then one of the solutions has $X \sim \omega^p$ with $p > -1/2$. For such a case,

$$\int_{-\infty}^{\infty} d\omega\, |\omega^{2p}| \propto \left[|\omega^{2p+1}|\right]_{-\infty}^{\infty}$$

which is infinity for $2p+1>0$, i.e. for $p > -1/2$. Hence, $D_M < 1/4$ will cause an unphysical solution (meaning the plasma is stable).

9. If $D_M \geq 1/4$ then

$$\omega^p = \omega^{-1/2}\,\omega^{\pm i|D_M-1/4|} = \omega^{-1/2}\,\exp\left[\ln\left(\omega^{\pm i|D_M-1/4|}\right)\right] = \omega^{-1/2}\,\exp\left[\pm i|D_M-1/4|\ln\omega\right].$$

For convergence at the infinite limits of the integral associated with the energy for the case $D_M = 1/4$ the two solutions have to cancel for large $\theta$ (to cancel the logarithmic singularity). Hence we require $A=-B$. One can think of the total solution as a delta function in $\theta$ (ideal MHD eigenfunctions always have a singularity at marginal stability). With the addition of a very small amount of inertia, growth rates can be calculated, and we will be able to resolve the singularity for small and large $\theta$. The two solutions will ensure finite energy.

10. with the earlier definition of $e$ we obtain

$$\hat{e} = \alpha s(\cos\omega - 1).$$

Thus we obtain (noting $c = s^2$ is a constant in $\omega$, so that $\langle 1/c\rangle = 1/s^2$ ) ,

$$\langle \hat{e}\rangle = -\alpha s, \quad \left\langle \frac{\hat{e}}{c}\right\rangle = -\frac{\alpha}{s}, \quad \left\langle \frac{\hat{e}\hat{e}}{c}\right\rangle = \frac{3}{2}\frac{\alpha^2}{s^2}.$$

Note also that $a$ and $b$ don't appear in $D_M$. And

$$\langle d\rangle = \frac{\alpha^2}{2} - \epsilon\alpha\left(1 - \frac{1}{q^2}\right),$$

i.e. the sinusoidal terms in $d$ average out. Substituting all of this into $D_M$ then yields,

$$D_M = \frac{\alpha}{s^2}\epsilon\left(\frac{1}{q^2} - 1\right).$$

Writing the instability condition as $cD_M > \frac{1}{4}c$, then we have,

$$\epsilon\alpha\left(\frac{1}{q^2} - 1\right) > \frac{1}{4}s^2$$

for instability. It means that the average curvature multiplied by minus the pressure gradient must be larger than the stabilising effect from magnetic field line bending.

The average curvature includes the effect of curvature in a cylinder, and average (over the cross section) toroidal curvature. Assuming $\alpha > 0$, the effect of average total curvature is favourable in a torus if $q > 1$, unfavourable if $q < 1$. This is because the unfavourable cylindrical curvature wins for $q < 1$, while favourable average toroidal curvature wins if $q > 1$. So in a cylinder the plasma will always be unstable to interchange modes if the magnetic shear is not strong enough to overcome the interchange drive. And so will reverse field pinches (RFPs) be unstable, since despite being toroidal, they operate with $|q| \ll 1$ ($q$ reverses sign near the edge of the plasma, hence the name *reverse field pinch*. That is the reason for the poor success of RFP's. Even worse, the resistive extension of interchange modes (resistive interchange modes) lose the stabilising effect of magnetic shear, so cylinders and RFPs are unstable to resistive interchange everywhere for any value of positive $\alpha$.

The exact reverse of the last paragraph is the case if $\alpha$ is negative. This can occur if the plasma is subject to strong core impurity radiation. In fact the rare cases where ideal interchange modes have been observed in tokamaks are cases where there are strong impurity influxes. The modes appearing where $q > 1$, $\alpha$ negative and $s$ small. So, they have been observed in advanced scenarios, $q_{min} > 1$, in the region of $q_{min}$ where the shear is zero, and where the impurities have caused a local change in sign of $\alpha$ at $q_{min}$

11. Writing the interchange equation in the form,

$$\frac{\partial}{\partial\omega}\left[\omega^2\frac{\partial}{\partial\omega}X\right] + D_M X = 0$$

we would have $a = 0$, $b = 0$, $c = 1$, $d = D_M$ and $e = 0$. Then,

$$p = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - D_M}$$

with

$$D_M = \left\langle\frac{\hat{e}}{c}\right\rangle - \left\langle\frac{\hat{e}}{c}\right\rangle^2 + \left\langle\frac{1}{c}\right\rangle\left(\left\langle\frac{\hat{e}\hat{e}}{c}\right\rangle + \langle d\rangle - \langle\hat{e}\rangle\right) = \langle d\rangle = D_M.$$

We recover the interchange equation from the ballooning equation if we take $s\omega >> 1$ and $\sin\omega \to 0$ and $\cos\omega \to 0$, which is to be expected for the secular (large $\omega$) behaviour.

No, interchange modes can't be recovered from

$$\frac{\partial}{\partial\omega}\left[\left\{1 + (s\omega - \alpha\sin\omega)^2\right\}\frac{\partial}{\partial\omega}X\right] + \alpha\left[\cos\omega + \sin\omega(s\omega - \alpha\sin\omega)\right]X = 0.$$

because the crucial contribution associated with average curvature (Mercier term)

$$\epsilon\alpha\left(\frac{1}{q^2} - 1\right)$$

is missing.

The reduced equation is sufficient for $q >> 1$ because the Mercier term is small (and stabilising) compared to the other ballooning contributions, providing $s$ and $\alpha$ are not too small.

12. Ballooning modes are more unstable than interchange modes because of the effect of local magnetic shear, at least that is the case if the shear is not too large, and $\alpha$ is not too small. As discussed in the notes, the local magnetic shear is given by,

$$s_l(r,\omega)^2 = \left(\frac{r}{q_l}\frac{dq_l}{dr}\right)^2 = (s(r) - \alpha(r)\cos\omega + O(\epsilon))^2 \quad \text{for} \quad \Delta' \sim \epsilon \quad \text{and} \quad r\Delta'' = \alpha + O(\epsilon).$$

Interchange modes will be stable in the region where $q > 1$ providing that $\alpha > 0$, which is usually the case (unless strong impurity concentration causes hollow pressure profiles). Ballooning modes can be unstable in the region $q > 1$ since $s_l^2$ vanishes locally on the outboard side ($\omega = 0$) for $\alpha \sim s$. The second region of stability is associated with very weak magnetic shear, and large $\alpha$, since it is seen that $s_l^2$ can become large and stabilising on the outboard side in that case. Operating in stable conditions with strong pressure gradients is clearly attractive, but it is difficult to obtain experimentally, because on the way towards obtaining the stabilising effect of large $\alpha$ clearly $\alpha$ will start from a smaller destabilising value. So rapid application of heating might help, or local current drive or shape control to adjust the magnetic shear or local shaping on the route to second region of stability.

It has been shown that the second region of stability can be accessed with very large triangularity and elongation. An extreme case is the bean-shape, but it can be quite unstable to $n = 0$ vertical instabilities. Spherical tokamaks also tend to operate with large elongation, large triangularity, and large $\alpha$. MAST and NSTX have had some success in this respect (also the START tokamak, which was the predecessor of MAST).