Lecture 6

Localised toroidal instabilities: ballooning modes and interchange modes

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Content of this lecture

6. Localised toroidal instabilities: ballooning modes and interchange modes

Perpendicular perturbed field components

Suitable choice of coordinate system makes the stability analysis much simpler. These first few slides provide a recap of the properties seen in exercise 3 for the field components and the magnetic operator.

One can always write the equilibrium magnetic field in the "Clebsch" form (see last lecture):

$$\boldsymbol{B} = \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla}\psi \tag{6.1}$$

where ψ is the poloidal flux, and we recall that in flux coordinates, $\boldsymbol{\nabla}\psi = \psi' \boldsymbol{\nabla} r$. Thus we see that the field is perpendicular to the ψ direction, and to the β direction.

Consider now the magnetic field perturbations according to Eq. (3.3),

$$\boldsymbol{\delta B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}).$$

Now consider the radial component of $\boldsymbol{\delta B}$:

$$\delta B^r \equiv \boldsymbol{\delta B} \cdot \boldsymbol{\nabla} r = \boldsymbol{\nabla} r \cdot \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}).$$

Employing the vector identity $\boldsymbol{\nabla} r \cdot (\boldsymbol{\nabla} \times \boldsymbol{D}) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} r \times \boldsymbol{D}) + \boldsymbol{D} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} r)$ and noting that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} r) = 0$ we have

$$\delta B^r = \boldsymbol{\nabla} \cdot [(\boldsymbol{\xi} \times \boldsymbol{B}) \times \boldsymbol{\nabla} r] = \boldsymbol{\nabla} \cdot [\boldsymbol{B}(\boldsymbol{\nabla} r \cdot \boldsymbol{\xi}) - \boldsymbol{\xi}(\boldsymbol{\nabla} r \cdot \boldsymbol{B})]$$

Moreover, since $\boldsymbol{\nabla} r \cdot \boldsymbol{B} = 0$ then

$$\delta B^r = \boldsymbol{\nabla} \cdot [\boldsymbol{B}(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} r)] = (\boldsymbol{\xi} \cdot \boldsymbol{\nabla} r)\boldsymbol{\nabla} \cdot \boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{\nabla}(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} r)$$

and since $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ then we finally obtain

$$\delta B^r = \boldsymbol{B} \cdot \boldsymbol{\nabla}(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} r) = \boldsymbol{B} \cdot \boldsymbol{\nabla}\xi^r \quad \text{, with } \xi^r \equiv \boldsymbol{\xi} \cdot \boldsymbol{\nabla} r$$

The magnetic operator

The important operator $\boldsymbol{B}\cdot\boldsymbol{\nabla}$ is known as the Magnetic Operator. Note that we haven't yet employed the Clebsch field yet. We have only used the field identity $\boldsymbol{\nabla} r\cdot\boldsymbol{B}=0$. Assume the Clebsch field, which together with using $\boldsymbol{\nabla}\beta\cdot\boldsymbol{B}=0$ we obtain the other component of the perpendicular field

$$\boldsymbol{\delta B}\cdot\nabla\beta=\boldsymbol{B}\cdot\boldsymbol{\nabla}(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\beta).$$

We note that the total perturbed perpendicular field strength is

$$\delta B_{\perp}^{2}=(\delta B^{r})^{2}+\frac{(\boldsymbol{\delta B}\cdot\nabla\beta)^{2}}{(\boldsymbol{\nabla}\beta)^{2}}.$$

$\delta B_{\perp}^{2}$ will vanish on the rational surface except through magnetic shear contributions in $\boldsymbol{\nabla}\beta$.

Let us now consider the magnetic operator. Writing

$$\boldsymbol{\nabla}=(\boldsymbol{\nabla}\psi)\frac{\partial}{\partial\psi}+(\boldsymbol{\nabla}\Theta)\frac{\partial}{\partial\Theta}+(\boldsymbol{\nabla}\phi)\frac{\partial}{\partial\phi} \tag{6.2}$$

where Θ is not necessarily the same as ω defined in lecture 1. Now, choose to apply the form $\boldsymbol{B}=F\boldsymbol{\nabla}\phi+\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\psi$, so that

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}=F\boldsymbol{\nabla}\phi\cdot\left[(\boldsymbol{\nabla}\psi)\frac{\partial}{\partial\psi}+(\boldsymbol{\nabla}\Theta)\frac{\partial}{\partial\Theta}+(\boldsymbol{\nabla}\phi)\frac{\partial}{\partial\phi}\right]+(\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\psi)\cdot\left[(\boldsymbol{\nabla}\psi)\frac{\partial}{\partial\psi}+(\boldsymbol{\nabla}\Theta)\frac{\partial}{\partial\Theta}+(\boldsymbol{\nabla}\phi)\frac{\partial}{\partial\phi}\right].$$

Writing the Jacobian of the (ψ,Θ,ϕ) system as $\mathcal{J}_{\psi,\Theta}=(\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\Theta)^{-1}$, and recalling that $\boldsymbol{\nabla}\phi^{2}=R^{-2}$ then

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}=\frac{F}{R^{2}}\left[\frac{\partial}{\partial\phi}+\frac{R^{2}}{F\mathcal{J}_{\psi,\Theta}}\frac{\partial}{\partial\Theta}\right].$$

The magnetic operator

Now, what is $R^2/(F\mathcal{J}_{\psi,\Theta})$? From lecture 1, one finds the exact relation

$$q_l(\psi,\theta) \equiv \frac{d\phi}{d\Theta} = \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\phi}{\boldsymbol{B}\cdot\boldsymbol{\nabla}\Theta} = \frac{F\mathcal{J}_{\psi,\Theta}}{R^2},$$

so that

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} = \frac{F}{R^2}\left[\frac{\partial}{\partial\phi} + \frac{1}{q_l}\frac{\partial}{\partial\Theta}\right] = \frac{1}{\mathcal{J}_{\psi,\Theta}}\left[q_l\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\Theta}\right].$$

It will now be clear that it is advantageous to choose a straight field line system (with convention $\Theta \to \theta$), for which (see chapter 2)

$$\mathcal{J}_{\psi,\theta}(\psi,\theta) = \frac{q(\psi)R(\psi,\theta)^2}{F(\psi)} \quad \text{and thus} \quad q_l = q(\psi)$$

In this section we are looking at ballooning and interchange modes, which are driven by pressure, and as such, depend crucially on toroidicity. It is reminded that in a toroidal system, linear MHD modes do not have a single poloidal mode number. In general we have that,

$$\boldsymbol{\xi} = \sum_{m=-\infty}^{\infty} \boldsymbol{\xi}^{(m)}(r)\exp(in\phi - im\theta - i\omega t). \tag{6.3}$$

For ballooning modes, the poloidal spectrum is in practice very large (we will come back to this later).

The poloidal spectrum of the perturbation is minimised for a straight field line system, as the magnetic operator has the simplest form. Unstable modes have weak field line bending stabilisation. For example, $|\delta B^r|^2 = 0$ for $\boldsymbol{B}\cdot\boldsymbol{\nabla}\xi^r = 0$, i.e.

$$\left[\frac{\partial}{\partial\phi} + \frac{1}{q(\psi)}\frac{\partial}{\partial\theta}\right]\xi^r = 0 \quad \text{which infers} \quad \xi^r = \hat{\xi}^r(r)\exp[in\phi - inq(r)\theta - i\omega t].$$

The magnetic operator

In fact, as will be seen, treating the full high n ballooning problem yields (consideration of full field line bending stabilisation, verses pressure drive),

$$\xi^r = \hat{\xi}^r(r,\theta)\exp[in\phi - inq(r)\theta - i\omega t] \tag{6.4}$$

where $\hat{\xi}^r(r,\theta)$ is a weak function of θ. Equations (6.3) and (6.4) will be reconciled with each other later, and both representations include toroidicy and mode coupling.

Let us see what happens if we don't choose a straight field line system. Choose e.g. the system employed in lecture 2, so that $\mathcal{J}_{\psi,\Theta} \to \mathcal{J}_{\psi,\omega}$, where $\mathcal{J}_{\psi,\omega}$ is defined in Eq. (2.11), and R given by Eq. (1.24). It is thus clear that,

$$q_l(r,\omega) \approx q(r)\left\{1 - (\epsilon + \Delta')\cos\omega + \sum_{m=2}^{\infty}\left(S'_m - (m-1)\frac{S_m}{r}\right)\cos(m\omega) + O(\epsilon^2)\right\},$$

and thus the magnetic operator will be,

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} = \frac{F(r)}{R(r,\omega)^2}\left[\frac{\partial}{\partial\phi} + \frac{1}{q(r)}\left\{1 + (\epsilon+\Delta')\cos\omega - \sum_{m=2}^{\infty}\left(S'_m - (m-1)\frac{S_m}{r}\right)\cos(m\omega) + O(\epsilon^2)\right\}\frac{\partial}{\partial\omega}\right].$$

After developing the stability problem in a convenient way in straight field line coordinates, we will have to transform equations into (ψ,ω,ϕ) coordinates because we will need equilibrium toroidal effects to be included correctly (and it was necessary to expand the equilibrium in (ψ,ω,ϕ) coordinates). After these considerations, we now commence the formal derivation of infinite n ballooning modes.

Eikonal representation for field aligned instabilities

As we have seen, the most unstable modes will tend to be aligned to the field lines $m \approx nq$. We require that our localised modes vary strongly only perpendicularly to the field lines ($\boldsymbol{k}_\perp$ large), while varying slowly, on the scale of the machine size, along the field lines ($k_\parallel$ small). We implement this ordering by means of an eikonal representation for $\boldsymbol{\xi}_\perp$ [Ref. Connor, Hastie and Taylor, 1978], i.e. with separation of fast and slowly varying dependence:

$$\boldsymbol{\xi}_\perp(\psi, \theta, \phi) \rightarrow \boldsymbol{\xi}_\perp(\psi, \theta, \beta) = \hat{\boldsymbol{\xi}}_\perp(\psi, \theta) \exp(in\beta), \quad \text{with} \quad \boldsymbol{B} \cdot \boldsymbol{\nabla}\beta(\psi, \theta, \phi) = 0$$

It is found that the most unstable localised pressure driven modes have large n (infinite n are the most unstable in a static (non rotating) equilibrium). As a result, $\exp(in\beta)$ is a rapidly varying function across the field lines, in particular $\boldsymbol{k}_\perp = -i\boldsymbol{\nabla}\beta$.

Along the field lines $\boldsymbol{\xi}_\perp$ varies slowly: β will be exactly constant along the field lines, but there will be a slow variation along the field lines through the θ variation contained in $\hat{\boldsymbol{\xi}}_\perp$. In particular, $k_\parallel \boldsymbol{\xi}_\perp = \exp(in\beta)\boldsymbol{b} \cdot \nabla\hat{\boldsymbol{\xi}}_\perp = \exp(in\beta)(F/qR^2)\partial\hat{\boldsymbol{\xi}}_\perp/\partial\theta$. For the straight field line system already described, it is straightforward to show that

$$\beta = \phi - q(\psi)\theta$$

provides the correct Clebsch definition (Eq. (6.1)) of the field $\boldsymbol{B} = \boldsymbol{\nabla}\beta \times \boldsymbol{\nabla}\psi$ so that it is identical to $\boldsymbol{B} = F\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi$. See also the chapter on tearing modes and the helical field! Thus, also, the requirement $\boldsymbol{B} \cdot \boldsymbol{\nabla}\beta(\psi, \theta, \phi) = 0$ is obvious.

We are now ready to consider the potential energy δW of the internal plasma region. Consider just the perpendicular potential energy in the plasma

$$\delta W_\perp = \int d^3x \left[|\delta B_\perp|^2 + B^2 \,|\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}|^2 - 2(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} P)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}^*_\perp) - J_\parallel (\boldsymbol{\xi}^*_\perp \times \boldsymbol{b}) \cdot \boldsymbol{\delta B}_\perp \right] \tag{6.5}$$

Pressure driven short wavelength instabilities

With the eikonal representation of the perturbed field,

$$\boldsymbol{\delta B}_\perp = \exp(in\beta)\boldsymbol{\delta \hat{B}}_\perp \quad \text{with} \quad \boldsymbol{\delta \hat{B}}_\perp = \left[\boldsymbol{\nabla} \times (\hat{\boldsymbol{\xi}}_\perp \times \boldsymbol{B})\right]_\perp$$

the energy becomes

$$\begin{aligned}\delta W_\perp &= \frac{1}{2}\int d^3x \left[|\delta B_\perp|^2 + B^2 \left|in\boldsymbol{\nabla}\beta \cdot \hat{\boldsymbol{\xi}}_\perp + \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\xi}}_\perp + 2\hat{\boldsymbol{\xi}}_\perp \cdot \boldsymbol{\kappa}\right|^2 - 2(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} P)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}^*_\perp) - \right. \\ & \left. J_\parallel (\boldsymbol{\xi}^*_\perp \times \boldsymbol{b}) \cdot \boldsymbol{\delta B}_\perp \right].\end{aligned}$$

We note at this point something of concern: the toroidal wavenumber, n, still appears in the field compression term $B^2 \left|in\boldsymbol{\nabla}\beta \cdot \hat{\boldsymbol{\xi}}_\perp + \boldsymbol{\nabla} \cdot \hat{\boldsymbol{\xi}}_\perp + 2\hat{\boldsymbol{\xi}}_\perp \cdot \boldsymbol{\kappa}\right|^2$, and with large n, this would lead to a massive stabilising energy contribution $(n^2/2)\int d^3x\, B^2|\boldsymbol{\nabla}\beta \cdot \boldsymbol{\xi}_\perp|^2$. In order to keep this term finite, the perturbation must be of the form

$$\boldsymbol{\xi}_\perp = \boldsymbol{\xi}_{\perp 0} + \frac{\boldsymbol{\xi}_{\perp 1}}{n}, \quad \text{with} \quad \boldsymbol{\xi}_{\perp 0} = \frac{X}{B}\boldsymbol{b} \times \boldsymbol{\nabla}\beta$$

where $X(\psi, \theta)$ is a scalar function (a stream function) which is independent of $\boldsymbol{\xi}_{\perp 1}$. The term of concern $in\boldsymbol{\nabla}\beta \cdot \boldsymbol{\xi}_\perp$ is now finite even for infinite n, since $\boldsymbol{\xi}_{\perp 0}$ is perpendicular to $\boldsymbol{\nabla}\beta$. Meanwhile, taking the infinite limit of n enables $\boldsymbol{\delta B}_\perp$ to be written in a simplified form (independent of $\boldsymbol{\xi}_{\perp 1}$):

$$\begin{aligned}\boldsymbol{\delta B}_\perp &= \{\boldsymbol{\nabla} \times [(X\boldsymbol{b} \times \boldsymbol{\nabla}\beta) \times \boldsymbol{b}]\}_\perp \\ &= \{\boldsymbol{\nabla} \times (X\boldsymbol{\nabla}\beta)\}_\perp \equiv \boldsymbol{\nabla} \times (X\boldsymbol{\nabla}\beta) - \boldsymbol{b}[\boldsymbol{b} \cdot \boldsymbol{\nabla} \times (X\boldsymbol{\nabla}\beta)] \\ &= (\boldsymbol{b} \cdot \boldsymbol{\nabla}X)\boldsymbol{b} \times \boldsymbol{\nabla}\beta.\end{aligned}$$

EPFL

The $J_\parallel$ term vanishes for infinite n because $\boldsymbol{\xi}^*_{\perp 0} \times \boldsymbol{b} = (X^*/B)\boldsymbol{\nabla}\beta$, which is clearly perpendicular to the recent definition of $\boldsymbol{\delta B}_\perp$ (and for infinite n clearly $\boldsymbol{\xi}_{\perp 1}$ vanishes). Consequently we obtain the energy

$$\begin{aligned}\delta W \quad &= \quad \frac{1}{2}\int d^3x \left[\frac{1}{B^2}|\boldsymbol{B}\cdot\boldsymbol{\nabla}X|^2(\boldsymbol{b}\times\boldsymbol{\nabla}\beta)^2 + \right.\\ &\left. B^2\left|i\boldsymbol{\nabla}\beta\cdot\hat{\boldsymbol{\xi}}_{\perp 1} + \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\xi}}_{\perp 0} + 2\hat{\boldsymbol{\xi}}_{\perp 0}\cdot\boldsymbol{\kappa}\right|^2 - \frac{2}{B^4}(\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}P)(\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\kappa})|X|^2\right].\end{aligned}$$

Now, $\boldsymbol{\xi}_{\perp 1}$ appears only in the stabilising field compression term, and so we are free to minimise this term with respect to $\boldsymbol{\xi}_{\perp 1}$. The term is minimised to zero with $\boldsymbol{\nabla}\beta\cdot\hat{\boldsymbol{\xi}}_{\perp 1} = i(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\xi}}_{\perp 0} + 2\hat{\boldsymbol{\xi}}_{\perp 0}\cdot\boldsymbol{\kappa})$. Finally, we see some more simplifications: since $\boldsymbol{\nabla}\beta$ is perpendicular to $\boldsymbol{B}$ and $\boldsymbol{B}\cdot\boldsymbol{\nabla}X = \mathcal{J}^{-1}_{\psi,\theta}\partial X/\partial\theta$, we have,

$$\delta W = \pi\int d\psi\, d\theta \mathcal{J}_{\psi,\theta}\left[\left(\frac{\boldsymbol{\nabla}\beta}{B\mathcal{J}_{\psi,\theta}}\right)^2\left|\frac{\partial X}{\partial\theta}\right|^2 - 2\left(\frac{\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}P}{B^2}\right)\left(\frac{\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\kappa}}{B^2}\right)|X|^2\right]. \tag{6.6}$$

Let us now address and simplify the second term in δW, the so called interchange term. Employing the Clebsch field:

$$\begin{aligned}\frac{\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}P}{B^2} \quad &= \quad \frac{[(\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi)\times\boldsymbol{\nabla}\beta]\cdot\boldsymbol{\nabla}P}{(\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi)\cdot(\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi)}\\ &= \quad \frac{\left[(\boldsymbol{\nabla}\beta)^2\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)\boldsymbol{\nabla}\beta\right]\cdot\boldsymbol{\nabla}P}{(\boldsymbol{\nabla}\beta)^2(\boldsymbol{\nabla}\psi)^2 - (\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)^2}\\ &= \quad \frac{dP}{d\psi} \quad \text{by employing} \quad \boldsymbol{\nabla}P = \boldsymbol{\nabla}\psi\,\frac{dP}{d\psi}\end{aligned}$$

The weird curvature

Moreover, the following quantity is fondly known as the 'weird' (w) component of the curvature:

$$\kappa_w = \frac{\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\kappa}}{B^2} = \frac{[(\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi)\times\boldsymbol{\nabla}\beta]\cdot\boldsymbol{\kappa}}{(\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi)\cdot(\boldsymbol{\nabla}\beta\times\boldsymbol{\nabla}\psi)} = \frac{\Big[(\boldsymbol{\nabla}\beta)^2\boldsymbol{\nabla}\psi-(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)\boldsymbol{\nabla}\beta\Big]\cdot\boldsymbol{\kappa}}{(\boldsymbol{\nabla}\beta)^2(\boldsymbol{\nabla}\psi)^2-(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)^2}.$$

From force balance the curvature vector is (see exercise series 3):

$$\boldsymbol{\kappa}\equiv(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b} = \left(\frac{1}{B^2}\right)[\boldsymbol{\nabla}-\boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})]\left(\frac{B^2}{2}+P(\psi)\right) \quad \text{where} \quad \boldsymbol{b}=\frac{\boldsymbol{B}}{B}.$$

We do not need to worry about subtracting the parallel derivative in B^2 from $\boldsymbol{\nabla}$ because the required operation on $\boldsymbol{\kappa}$ in $\boldsymbol{B}\times\boldsymbol{\nabla}\beta\cdot\boldsymbol{\kappa}$ is perpendicular to $\boldsymbol{B}$. So we can use for general flux coordinate system (ψ,Θ,ϕ):

$$\boldsymbol{\nabla}\left(\frac{B^2}{2}+P\right) = \left(\boldsymbol{\nabla}\psi\frac{\partial}{\partial\psi}+\boldsymbol{\nabla}\Theta\frac{\partial}{\partial\Theta}\right)\left(\frac{B^2}{2}+P\right)$$

which gives,

$$\kappa_w = \frac{1}{B^2}\left[\frac{[(\boldsymbol{\nabla}\beta)^2(\boldsymbol{\nabla}\psi)^2-(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)^2]\frac{\partial}{\partial\psi}+[(\boldsymbol{\nabla}\beta)^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\Theta-(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\Theta)]\frac{\partial}{\partial\Theta}}{(\boldsymbol{\nabla}\beta)^2(\boldsymbol{\nabla}\psi)^2-(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)^2}\right]\left[\frac{B^2}{2}+P\right]. \tag{6.7}$$

Local potential energy δW_ψ

And therefore we have

$$\kappa_w = \left(\frac{1}{B^2}\right)\left[\frac{\partial}{\partial\psi} + \left\{\frac{(\nabla\beta)^2\nabla\psi\cdot\nabla\Theta - (\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\Theta)}{B^2}\right\}\frac{\partial}{\partial\Theta}\right]\left(\frac{B^2}{2} + P\right). \quad (6.8)$$

The first term in square brackets is the radial curvature, while the second term is small correction proportional to the geodesic curvature, arising in Eq. (6.8) because of non-orthogonality in θ and ψ (which is small but present for straight field line coordinates and the analytic expanded equilibrium coordinates described in lecture 2). Substituting κ_w into δW we finally have the compact expression:

$$\delta W = \pi\int d\psi\, d\theta \mathcal{J}_{\psi,\theta}\left[\left(\frac{\nabla\beta}{B\mathcal{J}_{\psi,\theta}}\right)^2\left|\frac{\partial X}{\partial\theta}\right|^2 - 2\kappa_w\frac{dP}{d\psi}|X|^2\right]. \quad (6.9)$$

There are some important properties to note at this point. The expression for δW, and the corresponding, forthcoming, Euler equation, can undergo simple redefinition of the poloidal coordinate, since $\mathcal{J}_{\psi,\theta}d\theta$ and $\mathcal{J}_{\psi,\theta}^{-1}\partial/\partial\theta$ is independent of the poloidal coordinate. Note that when transforming to a new angle, quantities such as κ_w and β must be defined in terms of this new angle (exercises!).

Moreover δW contains only one dependent variable X, and one independent variable θ. The problem is one dimensional, in particular there are no radial derivatives. We can therefore consider a potential energy functional on each flux surface separately:

$$\delta W_\psi = \int d\Theta \mathcal{J}_{\psi,\Theta}\left[\left(\frac{\nabla\beta}{B\mathcal{J}_{\psi,\Theta}}\right)^2\left|\frac{\partial X}{\partial\Theta}\right|^2 - 2\kappa_w\frac{dP}{d\psi}|X|^2\right]. \quad (6.10)$$

Limits of integration in the arbitrary angle Θ will be considered later, as there are technicalities that will need to be properly discussed.

Variation of δW_ψ

One can now minimize δW_ψ with respect to X in order to assess the stability threshold (energy principle). Euler Lagrange equation for X defines the full poloidal dependence in X.

$$\frac{1}{\mathcal{J}_{\psi,\Theta}}\frac{\partial}{\partial\Theta}\left[\left(\frac{\nabla\beta}{B}\right)^2\frac{1}{\mathcal{J}_{\psi,\Theta}}\frac{\partial}{\partial\Theta}X\right]+2\kappa_w\frac{dP}{d\psi}X=0, \quad (6.11)$$

where again we note the independence of the definition of the poloidal angle Θ (but remembering that $\beta=\beta(\theta)$ etc).

The problem can be solved only if the equilibrium quantities (B, κ_w etc) are known in terms of a suitable poloidal angle. In lectures 1 and 2 we obtained an equilibrium expansion that included toroidal and shaping effects analytically. The poloidal angle ω was neither a straight field line variable, nor orthogonal. If we use that variable in Eq. (6.11), so that we solve,

$$\frac{1}{\mathcal{J}_{\psi,\omega}}\frac{\partial}{\partial\omega}\left[\left(\frac{\nabla\beta}{B}\right)^2\frac{1}{\mathcal{J}_{\psi,\omega}}\frac{\partial}{\partial\omega}X\right]+2\kappa_w\frac{dP}{d\psi}X=0 \quad (6.12)$$

we need a transformation between ω and straight field line angle θ, since e.g. $\beta=\phi-q(r)\theta$ must be obtained in terms of ω. For obtaining the transformation, we equate the volume element

$$d^3x=\mathcal{J}_{r,\omega}\,dr\,d\omega\,d\phi=\mathcal{J}_{\psi,\omega}\,d\psi\,d\omega\,d\phi=\mathcal{J}_{\psi,\theta}\,d\psi\,d\theta\,d\phi,$$

with $\mathcal{J}_{r,\omega}=\psi'\mathcal{J}_{\psi,\omega}$ giving

$$d\theta=d\omega\frac{\mathcal{J}_{\psi,\omega}}{\mathcal{J}_{\psi,\theta}}\quad\text{and}\quad\theta(\omega)=\int_\omega d\omega\,\frac{\mathcal{J}_\omega}{\mathcal{J}_{\psi,\theta}\psi'}.$$

Now the poloidal dependence in $\mathcal{J}_{\psi,\theta}$ is entirely contained in R^2, so that

$$\theta=\left(2\pi\Big/\int_0^{2\pi}d\omega\,\frac{\mathcal{J}_{\psi\omega}}{R^2}\right)\int_0^\omega d\omega\,\frac{\mathcal{J}_{\psi,\omega}}{R^2}.$$

Transformation to equilibrium coordinates

From the results of lecture 2 (Eq. (2.11)), we have for the analytic equilibria variables (see also exercise series 2):

$$\theta(\omega) = \omega - \varepsilon(\epsilon + \Delta')\sin\omega + \varepsilon \sum_{m=2}^{\infty} \frac{1}{m}\left(S'_m - (m-1)\frac{S_m}{r}\right)\sin(m\omega) + O(\varepsilon^2),$$

This change of coordinates is used inside the weird curvature of Eq. (6.8), which we may define as $\Theta \to \omega$:

$$\kappa_w = \left(\frac{1}{B^2}\right)\left[\frac{\partial}{\partial\psi} + \left\{\frac{(\nabla\beta)^2\nabla\psi\cdot\nabla\omega - (\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\omega)}{B^2}\right\}\frac{\partial}{\partial\omega}\right]\left(\frac{B^2}{2} + P\right). \quad (6.13)$$

Correct transformation needs to be made inside $\beta = \phi - q(\psi)\theta(\omega)$.

In the following calculation of the analytic ballooning equation we assume small pressure (conventional analytic assumption $\beta \sim \epsilon^2$), but pressure gradients are allowed to be large (appropriate for H-mode pedestal, or internal transport barrier). Specifically,

$$\alpha \equiv -\frac{2q^2 R_0}{B_0^2}\frac{dP}{dr} \sim O(\epsilon^0), \quad \text{while} \quad \frac{2P}{B_0^2} \sim O(\epsilon^2).$$

One then finds that

$$\Delta' \sim \epsilon, \quad \text{and} \quad r\Delta'' = \alpha + O(\epsilon^1)$$

so that, when developing the analytic ballooning equation, Δ' terms are dropped from the final expression, and $r\Delta''$ is replaced with α everywhere.

We drop shaping effects in what is to come, i.e. $S_m = 0$. Shaping effects do not appear in the leading order ballooning equation, nor in the ballooning diagram for standard shaping ordering $S_m/r \sim \epsilon$ if $s \sim \epsilon^0$ and $r\Delta'' \sim \alpha \sim \epsilon^0$. But shaping effects do modify interchange modes for $S_m/r \sim \epsilon$, $s \sim \epsilon$ and $r\Delta'' \sim \alpha \sim \epsilon$.

Transformation to equilibrium coordinates

Results required in the expansion of the ballooning equation (keeping toroidal effects, neglecting shaping effects) are (exercises!)

$$\nabla\beta = \nabla r\frac{\partial\beta}{\partial r} + \nabla\omega\frac{\partial\beta}{\partial\omega} + \nabla\phi\frac{\partial\beta}{\partial\phi} \quad \text{giving} \quad \nabla\beta = \nabla\phi + a\nabla r + b\nabla\omega$$

with

$$\beta \equiv \phi - q(\psi)\,\theta(\omega) \quad \text{where} \quad \theta(\omega) = \omega - (\epsilon + \Delta')\sin\omega. \quad \text{Thus:}$$

$$a = \frac{\partial\beta}{\partial r} = q\sin\omega\frac{1}{r}(\epsilon + r\Delta'') - \frac{q}{r}s[\omega - (\epsilon+\Delta')\sin\omega] \quad \text{and} \quad b = \frac{\partial\beta}{\partial\omega} = -q[1-(\epsilon+\Delta')\cos\omega],$$

with $s = (r/q)dq/dr$, giving,

$$(\nabla\beta)^2 = (\nabla\phi)^2 + a^2(\nabla r)^2 + b^2(\nabla\omega)^2 + 2ab\nabla r\cdot\nabla\omega = \frac{q^2}{r^2}\left[1 + \left(s\omega - r\Delta''\sin\omega\right)^2 + O(\epsilon\sin\omega, \epsilon\cos\omega)\right]. \tag{6.14}$$

In the definition of κ_w of Eq. (6.13) we require

$$(\nabla\beta)^2\nabla\psi\cdot\nabla\omega - (\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\omega) = \left[(\nabla\phi)^2\nabla r\cdot\nabla\omega + ab(\nabla r\cdot\nabla\omega)^2 - ab(\nabla r)^2(\nabla\omega)^2\right]\psi',$$

so that keeping only ϵ^0 and $\epsilon\sin\omega$ terms (knowing we will multiply by $\partial B^2/\partial\omega \approx 2B_0^2\epsilon\sin\omega$ in κ_w, knowing that $\epsilon\sin^2\omega^2$ will provide a contribution for interchange modes):

$$(\nabla\beta)^2\nabla\psi\cdot\nabla\omega - (\nabla\beta\cdot\nabla\psi)(\nabla\beta\cdot\nabla\omega) \approx -\psi' ab(\nabla r)^2(\nabla\omega)^2 = \frac{q^2(\psi')}{r^3}\left[(r\Delta'' + \epsilon)\sin\omega - s\omega + O(\epsilon, \epsilon\cos\omega\right.$$

Another quantity that is required in κ_w is

$$\boldsymbol{\kappa}\cdot\nabla r = \frac{1}{B^2}\frac{\partial}{\partial r}\left(\frac{B^2}{2} + P\right) \approx \frac{1}{B_0^2}\frac{dP}{dr} + \frac{R^2}{2R_0^2}\frac{\partial}{\partial r}\left[\left(\frac{R_0}{R}\right)^2\left(1 + 2F_2 + \frac{\epsilon^2}{q^2}(1 + 2\Delta'\cos\omega)\right)\right].$$

The ballooning equation

In addition (see lecture 2),

$$\frac{dF_2}{dr} = -\frac{1}{B_0^2}\frac{dP}{dr} - \frac{r}{R_0^2 q^2}(2-s) \ \text{ giving } \ \boldsymbol{\kappa}\cdot\boldsymbol{\nabla} r = \frac{1}{B^2}\frac{\partial}{\partial r}\left(\frac{B^2}{2}+P\right) = \kappa_{Rr} + \kappa_{rr}$$

where we break these terms down respectively as toroidal (κ_{Rr}) and poloidal (κ_{rr}) curvature contributions projected in the direction of the minor radius $\kappa_{Rr} = \boldsymbol{\kappa}_R \cdot \boldsymbol{\nabla} r$:

$$\kappa_{Rr} \approx \frac{R^2}{2}\frac{\partial}{\partial r}\left(\frac{1}{R}\right)^2 = -\frac{1}{R}\frac{\partial R}{\partial r} \approx -\frac{1}{R_0}\left(\cos\omega - \epsilon\cos^2\omega\right)$$

and

$$\kappa_{rr} = -\frac{\epsilon}{R_0 q^2}(1 - r\Delta''\cos\omega).$$

Also required in the weird curvature is:

$$\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\omega \approx \left(\frac{1}{r^2}\right)\frac{1}{2B^2}\frac{\partial B^2}{\partial\omega} \approx \frac{1}{rR_0}\sin\omega$$

Combining with $(\boldsymbol{\nabla}\beta)^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega - (\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\omega)$, makes a crucial contribution to κ_w, noting that $\epsilon(cos^2\omega + \sin^2\omega) = \epsilon$:

$$\kappa_w = -\frac{1}{\psi' R_0}\left[\cos\omega - \epsilon\left\{1 - \frac{1}{q^2}\right\} + \sin\omega(s\omega - r\Delta''\sin\omega) + O(\epsilon\sin\omega, \epsilon\cos\omega)\right]. \qquad (6.15)$$

By assuming the correct definition of κ_w given by Eq. (6.15), and using Eq. (6.14) for $(\boldsymbol{\nabla}\beta)^2$ one now easily obtains (from Eq. (6.12)) the large aspect ratio tokamak ballooning equation with circular cross section by also noting that $\psi' = rB_0/q$:

$$\frac{\partial}{\partial\omega}\left[\left\{1 + (s\omega - \alpha\sin\omega)^2\right\}\frac{\partial}{\partial\omega}X\right] + \alpha\left[\cos\omega - \epsilon\left\{1 - \frac{1}{q^2}\right\} + \sin\omega(s\omega - \alpha\sin\omega)\right]X = 0. \qquad (6.16)$$

Notes

We note that in a cylinder (screw pinch approximation) we do not have the toroidal curvature (κ_R), so that

$$\boldsymbol{\kappa} \cdot \boldsymbol{\nabla} r(cyl) = \kappa_{rr}(\text{with}\, \Delta'' = 0) = -\frac{\epsilon}{R_0 q^2}.$$

The consequence of having no toroidal curvature in a torus is that configuration becomes much more unstable. We will see that the Mercier criterion for unstable interchange modes in a torus $D_M > 1/4$ with $D_M = -\frac{\epsilon}{s^2}\alpha\left[1 - \frac{1}{q^2}\right]$ is replaced with the Suydam criterion for instability in a cylinder $D_S > 1/4$ with $D_S = -\frac{\epsilon}{s^2}\alpha\left[0 - \frac{1}{q^2}\right]$

The ballooning representation

EPFL

Let us consider the periodicity of X via considering $\exp(in\beta)$ where $\beta = \phi - \theta q(\psi)$. It is clear that

$$\exp(in\beta) = \exp[in\phi - inq(\psi)\theta]$$

is not periodic in θ except on a rational surface where $nq = m$ (with m integer). For example

$$\exp[in\beta(\theta + 2\pi, \phi)] = \{\cos[2\pi nq(\psi)] - i\sin[2\pi nq(\psi)]\}\ \exp[in\beta(\theta, \phi)].$$

The same lack of periodicity holds also for our analytic equilibrium coordinate ω. Since $\theta(\omega) = \omega - (\epsilon + \Delta')\sin\omega$, then $\theta(\omega + 2\pi) = \theta(\omega) + 2\pi$, and thus

$$\exp[in\beta(\omega + 2\pi, \phi)] = \exp[-i2\pi nq(\psi)]\ \exp[in\beta(\omega, \phi)].$$

Connor, Hastie and Taylor realised that we can give up on $X(\Theta)$ being periodic in Θ (i.e. in any angle e.g. θ or ω). They allowed the angle Θ to be a generalised coordinate, mapping out the entire length of a field line, from minus infinity, to plus infinity. Thus the solution of Eq. (6.11) is a variation of the modified local potential energy,

$$\delta W_{\psi,\omega} \propto \int_{-\infty}^{\infty} d\omega \mathcal{J}_{\psi,\omega} \left[\left(\frac{\boldsymbol{\nabla}\beta}{B\mathcal{J}_\omega} \right)^2 \left| \frac{\partial X}{\partial \omega} \right|^2 - 2\kappa_w \frac{dP}{d\psi} |X|^2 \right]. \tag{6.17}$$

Convergence of $\int_{-\infty}^{\infty} d\omega |X|^2$ requires that $X \sim |\omega|^{-1/2}$, or faster, as $\omega \to \pm\infty$. This forms the boundary condition for X, i.e. that the solution remains physical (finite energy). Stability boundaries are formed by solution of ballooning equation Eq. (6.11), and substitution into Eq. 6.17), varying equilibrium parameters searching for $\delta W_\psi = 0$.

Notes

A practical solution to the difficulty of non-periodicty, and a reconciliation of periodic solutions from the ballooning represented solutions is obtained by considering the following:

1. The linear second order differential equation of Eq. (6.16) has in general two independent solutions, one of which vanishes for $\omega \to -\infty$, the other $\omega \to +\infty$. Marginal stability corresponds to special values of the equilibrium quantities for which the equation has a solution that vanishes for $\omega \to \pm\infty$ simultaneously. This is required because the eigenmode must have a finite energy content.

2. The solution $\boldsymbol{\xi}_Q$ to Eq. (6.16) with extended angle is known as a quasi-solution. It is associated with X as follows:

$$\boldsymbol{\xi}_Q = \frac{X}{B}\boldsymbol{b} \times \boldsymbol{\nabla}\beta.$$

   As already shown, this eigenfunction is not periodic in ω, and hence it is not physical, but it is nevertheless a solution of Eq. (6.16) in the extended space. Furthermore, the general MHD force operator δF of Eq. (3.4), and the force in Eq. (6.16), is periodic. This means that,

$$\boldsymbol{\xi}_Q(\omega + 2\pi k) = \exp[-ik2\pi n q(\psi)]\boldsymbol{\xi}_Q(\omega)$$

   is also a solution of $\delta F = 0$ everywhere in the extended space.

3. The force operator is linear, which means that the following **periodic** sum satisfies $\delta F = 0$:

$$\xi(\psi, \omega, \phi) = \sum_{k=-\infty}^{\infty} \boldsymbol{\xi}_Q(\omega + 2\pi k).$$

According to observation 2, all terms in the sum satisfy the equation of motion at marginal stability and point 3 (linearity) guarantees that the sum, if it exists, also satisfies this equation. Observation 1 states that the necessary boundary conditions for the existence of the sum, $X \sim |\omega|^{-1/2}$ for $\omega \to \pm\infty$, are satisfied in the case of marginal stability. Finally, according to point 3, the sum defines a periodic solution of the marginal stability equation. We thus consider the sum the physical solution.

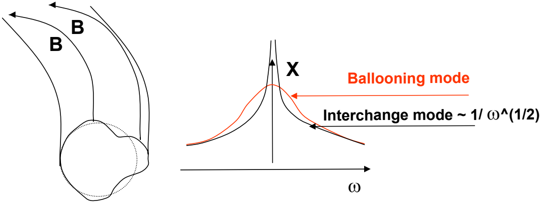
The solution ξ is a sum over many terms that have different values of the non-periodic function $\exp[-ik2\pi nq(\psi)]$, which contains the radial dependence of ξ. Note that the individual quasi-modes are not bounded in the radial direction since the exponent $\exp[-ik2\pi nq(\psi)]$ does not vanish anywhere. Fortunately it has been shown (not proven here) that the sum ξ is radially localized at the flux surface where we have solved Eq. (6.16)).

Ballooning structure

Consider the interchange contribution to the energy, i.e.

$$\int_{-\infty}^{\infty} d\omega \mathcal{J}_\omega \left(-2\kappa_w \frac{dP}{d\psi} |X|^2 \right).$$

The minimising solution for X takes a ballooning form. The main reason for this is that the local shear is weaker on the outboard side than it is on the inboard side (see next few slides), meaning that field line bending stabilisation is weaker on the outboard side, so the instability tends to be concentrated there.



In the ballooning representation there will be a long tail in ω ($X \sim |\omega|^{-1/2}$ for $\omega \to \pm\infty$) that is captured also by interchange modes, see later. Moving to physical periodic modes, we will require a large spectrum of modes to develop the ballooning structure, i.e. a mode dominantly on the outboard side. In fact there are an infinite number of significant harmonics for $n \to \infty$, indicating that the ballooning mode is a true toroidal instability, and it is hopeless to attempt truncation of the harmonics via an inverse aspect ratio expansion.

Ballooning stability diagram

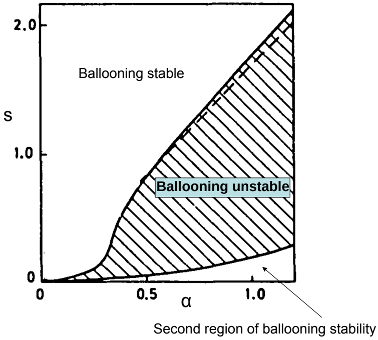
Analysis of infinite n ballooning modes provide a necessary and sufficient condition for stability of local (short wavelength) ideal internal MHD instabilities.

The diagram was created by solving Eq. (6.11), and substitution into Eq. 6.17), varying equilibrium parameters α and s, searching for $\delta W_\psi = 0$. The higher order Mercier term $-\epsilon\alpha(1 - 1/q^2)$ is not very significant in for ballooning diagram.

For the first stability boundary, shear is stabilising because it increases field line bending (see $\nabla\beta$ contribution to $|\delta B_\perp|^2$). While α increases ballooning/interchange drive.

For many years tokamaks have tried to operate in the second region if stability. It can be expanded with extreme elongation and triangularity (partly why spherical tokamaks are successful). It can be achieved with super H-mode operation. The reason for this effect is due to local shear which will be explained next.

Conventional pressure and q profiles chosen.
q>1, so Mercier stability everywhere



Effect of local magnetic shear on ballooning

It is clear that magnetic shear is stabilising. This is not surprising because, in order to minimise magnetic field line bending stabilisation, instabilities tend to align themselves with the equilibrium field on a given flux surface. The magnetic shear determines the rate at which the mode and the equilibrium magnetic field become misaligned on neighbouring flux surfaces.

A feature that is missing from the large ω treatment of interchange modes is the role of poloidally localised magnetic shear. These effects are taken care of in the general ballooning equation of Eq. (6.16) via the $\alpha \sin \omega$ term. In order to see the effect of local shear, we should first examine the local q defined above in Eq. (1.17), and in the last lecture:

$$q_l \equiv \frac{d\phi}{d\omega} \approx q(r) \left\{ 1 - (\epsilon + \Delta') \cos \omega \right\},$$

if circular (unshaped, $S_m = 0$) flux surfaces are assumed. Hence the square of the local shear

$$s_l(r,\omega)^2 = \left(\frac{r}{q_l} \frac{dq_l}{dr} \right)^2 = (s(r) - \alpha(r) \cos \omega + O(\epsilon))^2 \quad \text{for} \quad \Delta' \sim \epsilon \quad \text{and} \quad r\Delta'' = \alpha + O(\epsilon).$$

The stabilising effect of shear is reduced at $\omega = 0$ as α is increased from a small positive value. This ensures that ballooning modes bulge on the outboard side and are more unstable than interchange modes at small to moderate α. This local shear effect, and its impact on magnetic field line bending, is nullified (averages out) when taking the large ω assumption of interchange modes. Ballooning modes tend to be unstable in the edge of a tokamak where $q \gg 1$, $s \sim 1$ and $\alpha \sim 1$ (interchange are stable for these parameters).

By further increase in α, the local minimum in s_l^2, and the minimum in field line bending stabilisation, occurs at larger values of ω (outboard side becomes region of improved curvature). As a result, very large values of α and small global shear s can yield the second region of stability to the ballooning mode. In this parameter range, the ballooning mode has no pressure driven trigger. Extreme shaping can extend the possibility of second region of stability.

Interchange modes

Stability to interchange modes is usually considered a necessary condition for ideal MHD stability. They play a very important role in setting the stability boundary in

- stellarators
- reverse field pinches
- tokamaks in $q < 1$ region
- tokamaks in the $q > 1$ region if impurities cause a reverse in the sign of the pressure gradient.

Stability to interchange modes is concerned with the stability corresponding to the large ω behaviour of X in the ballooning representation. The result will usually be an underestimate of instability relative to solving the full ballooning problem. The procedure for solving the interchange problem is to

1. Solve the ballooning equation for large ω.
2. Assume that the large ω solution is valid for all ω. This identifies the eigenfunction everywhere.
3. Obtain the condition for marginal stability by imposing physical boundary conditions

Interchange modes

Simply write the ballooning equation (Eq. 6.16) in the form (note ϵ correction is crucial!):

$$\frac{d}{d\omega}\left[f\frac{dX}{d\omega}\right]+gX=0$$

and we let

$$f=a+b\omega+c\omega^2 \quad \text{and} \quad g=d+e\omega$$

where a, b, c, d and e are periodic functions of ω

It is the secular dependence (long trend in ω) that needs to be captured in the large ω solution of X. We also know that we require $X\sim\omega^{-1/2}$, or faster, for convergence. We write X as an expansion in a large variable (reciprocal Taylor expansion):

$$X=\omega^p\left[X_0(\omega)+\frac{X_1(\omega)}{\omega}+\frac{X_2(\omega)}{\omega^2}+O\left(\frac{X_3}{\omega^3}\right)\right]$$

where again, X_0, X_1 and X_2 are 2π periodic (no secular dependence in ω). We must solve for X_0, X_1 and X_2 by substitution into the ballooning equation, and examination of coefficients in the resulting power series. Substitution of X_0, X_1 and X_2 into the ballooning equation then yields the power index p.

The result is (exercise!)

$$X=\omega^p[1+O(1/\omega)] \quad \text{with} \quad p=-\frac{1}{2}\pm\sqrt{\frac{1}{4}-D_M}$$

where

$$D_M=\left\langle\frac{\hat{e}}{c}\right\rangle-\left\langle\frac{\hat{e}}{c}\right\rangle^2+\left\langle\frac{1}{c}\right\rangle\left(\left\langle\frac{\hat{e}\hat{e}}{c}\right\rangle+\langle d\rangle-\langle\hat{e}\rangle\right) \quad , \quad \hat{e}=\int_0^\omega d\omega\, e, \quad \text{and} \quad \langle X\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\omega\, X$$

EPFL

If well behaved solutions exist the system is ballooning unstable. We require that both of these are well behaved solutions (physical). Note that if $D_M < 1/4$, one of the solutions will give rise to $p > -1/2$, and hence the energy will be infinite.

On the other hand, if $D_M > 1/4$, then p is complex, and both solutions are well behaved:

$$\omega^p = \omega^{-1/2}\,\omega^{\pm i|D_M - 1/4|}.$$

This result is oscillatory in D_M (and in r or ψ as we postulated earlier), since

$$\omega^p = \omega^{-1/2}\,\omega^{\pm i|D_M - 1/4|} = \omega^{-1/2}\,\exp\left[\ln\left(\omega^{\pm i|D_M - 1/4|}\right)\right] = \omega^{-1/2}\,\exp\left[\pm i|D_M - 1/4|\ln\omega\right].$$

Substituting for the coefficients (see Eq. 6.16) $a = 1 + \alpha^2\sin^2\omega$, $b = -2\alpha s\sin\omega$, $c = s^2$, $d = \alpha[\cos\omega - \epsilon[1 - 1/q^2)] - \alpha\sin^2\omega]$ and $e = \alpha s\,\sin\omega$ one ends up with

$$D_M = \frac{\alpha}{s^2}\epsilon\left(\frac{1}{q^2} - 1\right).$$

The condition for instability is $D_M > 1/4$ or

$$\alpha\epsilon\left(\frac{1}{q^2} - 1\right) > \frac{1}{4}s^2$$

which is the condition that pressure driven (average curvature driven) interchange can overcome magnetic field line bending stabilisation under the ideal MHD model.

Interchange modes and average curvature

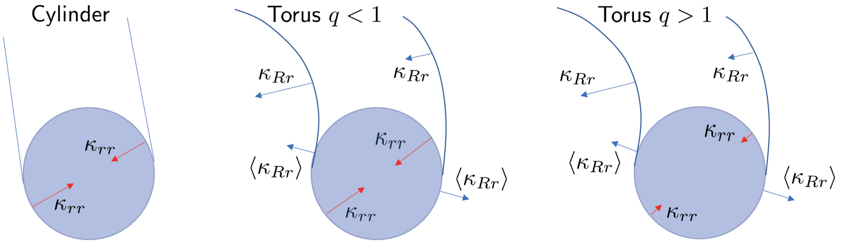
It will be shown in the exercises that we obtain the same result (i.e. the same $X = \omega^p$) for the following equation:

$$\frac{\partial}{\partial \omega}\left[\omega^2 \frac{\partial}{\partial \omega} X\right] + D_M X = 0 \quad \text{with} \quad D_M = -\frac{\epsilon}{s^2}\alpha\left[1 - \frac{1}{q^2}\right]. \tag{6.18}$$

This is the ballooning equation Eq. (6.16) with $s\omega >> 1$ and $\sin\omega \to 0$ and $\cos\omega \to 0$, i.e. Eq. (6.18) represents the secular limit of Eq. (6.16).

In a torus, the net curvature is favourable (stabilising) if $q > 1$ and $dP/dr < 0$ (conventional pressure gradient). This threshold changes with shaping.

In a cylinder $D_M \to D_S = -\frac{\epsilon}{s^2}\alpha\left[0 - \frac{1}{q^2}\right]$, we lose effect of stabilising toroidal curvature.



Curvature inwards (destabilising)

Outward average toroidal curvature $\langle\kappa_{Rr}\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\omega\, \kappa_{Rr}$ But net average curvature is inward (destabilising)

Average toroidal curvature $\langle\kappa_{Rr}\rangle$ outward. Weak cylindrical curvature κ_r. Net average curvature outward (stabilising)