Lecture 4

External kink modes and inertia treatment for ideal and resistive problems

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Content of this lecture

4. External kink modes and inertia treatment for ideal and resistive problems

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- Constant-psi approximation and asymptotic matching

Linear MHD equations

Now that we have learned that ideal internal modes are stable to order ϵ^2, we need to look more carefully at the vacuum-plasma interface, and the vacuum region itself. We will see that, under certain conditions, modes that extend up to and beyond the plasma edge can be unstable.

To treat external modes, we have to consider the plasma surface terms in the second term of Eq. (3.33)

$$\frac{2\pi^2 B_0^2}{R_0}\left\{a^2 \xi_0^r(a)^2\left[\frac{2}{q_a}\left(\frac{n}{m}-\frac{1}{q_a}\right)+\left(\frac{n}{m}-\frac{1}{q_a}\right)^2\right]\right\},$$

as well as the vacuum term, defined in Eq. (3.16) given by

$$\delta W_V = \frac{1}{2}\int_V d^3x \left|\boldsymbol{\delta\hat{B}}\right|^2 .$$

Equation (3.33) is a valid representation of δW_P to order ϵ^2. Moreover, the minimisation of the plasma region remains valid, so that Eq. (3.34) holds for the displacement. Indeed, variation of δW_P with respect to ξ_0^r is necessarily a variation of just the first term of Eq. (3.33), since the second term of Eq. (3.33) is simply proportional to the square of the displacement at the plasma edge, $\xi_0^r(a)$. Notice that the current that drives external kink modes, visible in δW_P (last term in Eq. (3.20)) is visible (involving $d/dr(r^2/q)$) in the second line of the equation above Eq. (3.33), which becomes the drive in the boundary terms in the second line of Eq. (3.33)

It remains to obtain δW_V by solving for the vacuum perturbed fields, subject to boundary conditions of Eq. (3.17) matching the internal minimised solution obtained from Eq. (3.34). Clearly, in the vacuum region there is no plasma, so the displacement is zero beyond the vacuum-plasma interface. Our goal is to obtain δW_V in terms of a common perturbed quantity appearing in δW_P. This could be the displacement at the plasma edge a, or a magnetic field component at a. We will choose $\xi_0^r(a)$.

External Kink Modes - Vacuum Region

Let us begin by looking at the perturbed parallel field in the plasma region. It can be shown that (Eq. (3.18) and exercise series 3):

$$\delta B_{\|} = \frac{\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} P}{B} - B(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}).$$

We note that $(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})$ has been minimised to zero, and moreover, $P/B^2 \sim \epsilon^2$, and near the edge, the pressure is vanishingly small. Hence, $\delta B_{\|} = 0$ at this order in the plasma region, and thus, due to the boundary condition of Eq. (3.17), remains zero in the vacuum region to relevant order. Since $\delta B_{\|} \approx \delta B^{\phi}$ (see also exercise week 3 for explicit calculation of δB^{ϕ}), we are left once again with δB^r and δB^{θ} in $|\boldsymbol{\delta B}|^2$.

The analysis is simplified by defining a perturbed poloidal flux $\delta\psi$ for which

$$\delta B^{\theta} = \frac{\partial \delta\psi}{\partial r} \tag{4.1}$$

(see the similar definition of the equilibrium poloidal flux of Eq. (1.10)). Applying $\boldsymbol{\nabla} \cdot \boldsymbol{\delta \hat{B}} = 0$ we have to leading order

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta B^r) + \frac{1}{r}\frac{\partial\,\delta B^{\theta}}{\partial \theta} = 0$$

and upon applying Eq. (4.1) this becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta B^r) = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial \delta\psi}{\partial r}\right)$$

and thus, since $\partial/\partial\theta = -im$, then in the vacuum:

$$\delta B^r = im\frac{\delta\psi}{r} \ \text{ giving } \ |\delta B|^2 = |\delta B^{\theta}|^2 + |\delta B^r|^2 = \frac{m^2}{r^2}\delta\psi^2 + \left(\frac{d\delta\psi}{dr}\right)^2. \tag{4.2}$$

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In our flux coordinates, we note that the boundary condition of Eq. (3.17) enforces the matching of δB^r on either side of the interface (flux label r is normal to all magnetic surfaces in the plasma, including at the edge). So equating at $r = a$ the vacuum description (Eq. (4.2)) of δB^r with that of the plasma description (Eq. (3.31)):

$$\delta\psi_a = \frac{B_0}{R_0}\left(\frac{n}{m} - \frac{1}{q_a}\right) a\,\xi_a \tag{4.3}$$

Let us now consider $\boldsymbol{\nabla} \times \boldsymbol{\delta\hat{B}} = 0$ (no currents in the vacuum), which in the ϕ direction gives

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta B^\theta) - \frac{1}{r}\frac{\partial\,\delta B^r}{\partial\theta} = 0$$

So, that, $\delta\psi$ satisfies Laplace equation on a circular disk:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\delta\psi}{dr}\right) - \frac{m^2}{r^2}\delta\psi = 0, \tag{4.4}$$

which has solution

$$\delta\psi = \alpha r^m + \beta r^{-m}. \tag{4.5}$$

Now substituting the BC at $r = a$ given by Eq. (4.3), and the BC $\delta\psi_b = 0$ at $r = b$ ($\delta B^r = 0$ at b as defined in Eq. (3.17)), eventually gives (see exercises):

$$\delta\psi = \frac{B_0}{R_0}\left(\frac{n}{m} - \frac{1}{q_a}\right)\frac{\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m}{\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m}\ a\,\xi_a$$

Returning to δW_V, and inserting Eq. (4.2), and integrating by parts (see exercises):

$$\delta W_V = 2\pi^2 R_0\left\{\int_a^b dr\, r\left[\frac{m^2}{r^2}\delta\psi^2 - \delta\psi\frac{1}{r}\frac{d}{dr}\left(r\frac{d\delta\psi}{dr}\right)\right] + \left(r\delta\psi\frac{d\delta\psi}{dr}\right)\Bigg|_a^b\right\}.$$

External Kink Modes

The first term is zero due to Laplace's equation (Eq. (4.4)), and inserting Eq. (4.5) into the second term we have the stabilising vacuum effect:

$$\delta W_V = \frac{2\pi^2 B_0^2}{R_0} m\lambda a^2 \xi_0^r(a)^2 \left(\frac{n}{m} - \frac{1}{q_a}\right)^2, \quad \text{with} \quad \lambda = \frac{1+(a/b)^{2m}}{1-(a/b)^{2m}}$$

which, combining with Eq. (3.33) gives the total potential energy including the internal plasma, the plasma-vacuum interface, and the vacuum terms:

$$\begin{aligned} \delta W \quad = \quad & \frac{2\pi^2 B_0^2}{R_0} \left\{ \int_0^a dr\, r \left[\left(r \frac{d\xi_0^r}{dr} \right)^2 + \left(m^2 - 1\right) (\xi_0^r)^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2 \right. \\ & \left. + \, a^2 \xi_0^r(a)^2 \left[\frac{2}{q_a} \left(\frac{n}{m} - \frac{1}{q_a} \right) + (1 + m\lambda) \left(\frac{n}{m} - \frac{1}{q_a} \right)^2 \right] \right\}. \end{aligned} \qquad (4.6)$$

No-wall limit $b/a \to \infty$ ($\lambda \to 1$ destabilising) and no-vacuum limit $b/a \to 1$ ($\lambda \to 1$ stabilising).

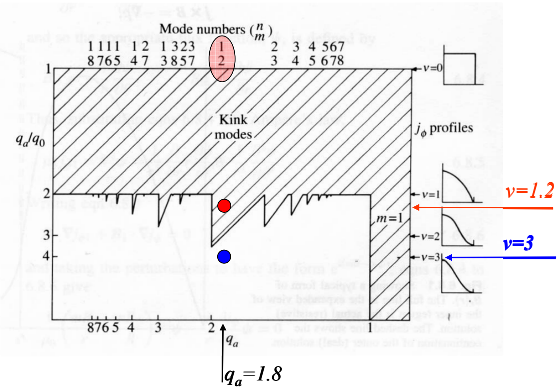
The mode is determined by the minimised δW in the plasma region, i.e. by Eq (3.34), which can be written in the convenient form

$$r^2 \frac{d^2 \xi_0^r}{dr^2} + r \frac{d\xi_0^r}{dr} \left[3 - \frac{2s(r)}{1 - \frac{n\, q(r)}{m}} \right] - (m^2 - 1)\xi_0^r = 0. \qquad (4.7)$$

The procedure is to insert the solution of Eq. (4.7) back into Eq. (4.6), and then determine the magnitude of the corresponding first term in δW, which is a measure of field line bending stabilisation integrated over the plasma volume. We will therefore determine whether this gross field line bending stabilisation, together with stabilising surface and vacuum terms, is sufficient to compensate the effects resulting from the destabilising surface terms in δW. It is clear that the surface contribution (the second term in Eq. (4.6) can be destabilising for $q_a < m/n$, i.e. it can be unstable when the rational $q = m/n$ is not inside the plasma region.

External Kink Modes

We will see that, strongest instability occurs when q_a is close to, but less than m/n. Vacuum effects are in the no-wall limit. Instability is stronger if global magnetic field line bending is weak, which is determined by $q_a/q_0 = 1 + \nu$ in Wesson diagram. These comments essentially explain Wesson's famous external kink stability diagram.



We investigate the red dot (unstable), and the blue dot (stable). Figure from Wesson, Tokamaks

We see that Wesson's diagram is in terms of
(1) mode numbers m and n
(2) Edge value of safety factor q_a
(3) Degree of peaking of the current profile (ν or q_a/q_0).
(4) The ratio b/a, noting that the most unstable situation is the no-wall limit $b/a \to \infty$, and the stable definite case is the vacuum-free case $a = b$ (perfectly conducting wall on the plasma edge).

Let us examine the most obvious and unstable (except for $m = n = 1$ external kink requiring $q_a < 1$) which is $m = 2$ and $n = 1$. As we have mentioned, instability requires $q_a < m/n$ Choose $q_a = 1.8$. Choose $b/a = 2$. Finally, examine, numerically, two cases from the stability diagram $\nu = 1.2$ (unstable) and $\nu = 3$ (stable). Wesson's current profile iso of the form $J = J_0[1 - (r/a)^2]^\nu$, which, from Eq. (3.32) is

$$q = r^2 \frac{B_0}{R_0} \Big/ \int_0^r dr\, r J_\phi \;=\; \frac{q_0(1+\nu)(r/a)^2}{1 - [1 - (r/a)^2]^{1+\nu}},$$

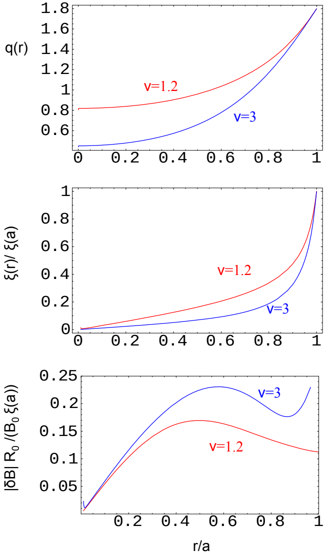
where $q_0 = q(r = 0) = 2B_0/(R_0 J_0)$ and it is seen that

$$\frac{q_a}{q_0} = 1 + \nu \quad \text{and} \quad q = \frac{q_a(r/a)^2}{1 - [1 - (r/a)^2]^{1+\nu}}.$$

We then insert the chosen values for ν, q_a, m, n and b/a and solve for the most unstable mode in the plasma region (solve Eq. 4.7). Then substitute the solution for $\xi_{r0}(r)/\xi_{r0}(a)$ into δW (Eq. 4.6) and evaluate the stabilising plasma contribution $\delta\hat{W}(plasma)$, and the destabilising external contributions $\delta\hat{W}(external)$. These contributions are the first and send lines of:

$$\delta\hat{W} = \frac{R_0 \delta W}{2\pi^2 a^2 \xi_0^r(a)^2 B_0^2} = \frac{1}{a^2}\int_0^a dr\, r \left[\left\{ r\frac{d}{dr}\left(\frac{\xi_0^r(r)}{\xi_0^r(a)}\right)\right\}^2 + \left(m^2 - 1\right)\left(\frac{\xi_0^r(r)}{\xi_0^r(a)}\right)^2\right]\left(\frac{n}{m} - \frac{1}{q}\right)^2$$
$$+ \frac{2}{q_a}\left(\frac{n}{m} - \frac{1}{q_a}\right) + (1 + m\lambda)\left(\frac{n}{m} - \frac{1}{q_a}\right)^2.$$

External kink stability with different equilibria



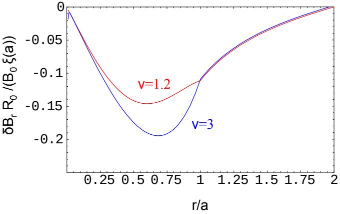
For $\nu = 3$, the enhanced global magnetic shear (q_a/q_0) leads to enhanced field line bending stabilisation $|\delta B_\perp|^2$.

$\nu = 1.2$: we have $\delta\hat{W}(plasma) = 0.0447$ and $\delta\hat{W}(external) = -0.0516461$. Hence $\delta W < 0$.

$\nu = 3$: we have $\delta\hat{W}(plasma) = 0.059671$ and $\delta\hat{W}(external) = -0.0516461$. Hence $\delta W > 0$.

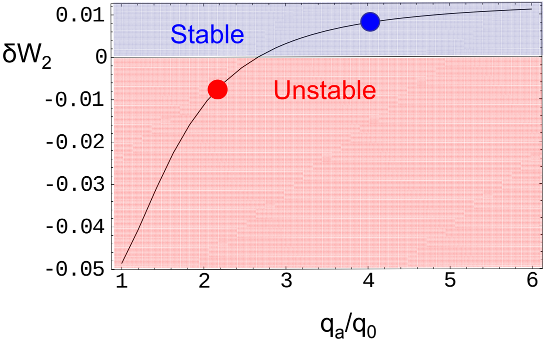
It is straightforward to evaluate the perturbed radial magnetic field in both regions

$$\frac{\delta B^r R_0}{B_0 \xi_{r0}(a)} = \begin{cases} \left(n - \dfrac{m}{q(r)}\right) \dfrac{\xi_0^r(r)}{\xi_0^r(a)} & \text{for } r \le a \\ \left(n - \dfrac{m}{q_a}\right) \dfrac{\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m}{\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m} & \text{for } a \le r \le b. \end{cases}$$



External kink stability with different equilibria

See exercises for how the growth rate can be approximated from the δW variational approach developed here .



Introduction to Inertia and Resistivity

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Magnetic shear usually effectively stabilises internal modes at order δW_2. But the $m = 1$ mode is an exception (see internal terms in δW_2).

It has been mentioned that in the region where magnetic field line bending is nullified, the ideal Euler equation for the plasma displacement, i.e. Eq. (3.34), is singular. More physics is needed in the region of $q = m/n$ in order to resolve the details of singularity.

The extra physics considered here comes in two forms: plasma inertia and resistivity. We note that inertia can be small, indeed vanishing small if sufficiently close to stability boundary (recall that $\delta K = -(\omega^2/2)\int |\boldsymbol{\xi}|^2 d^3x$ measures inertia), and moreover, resistive timescales (defined in a few pages) are slow compared to ideal timescales. Nevertheless, in the region of the rational surface, inertia and resistivity provide crucial corrections.

Inertia in the absence of resistivity is sufficient to resolve the singularity. But, as evident from the derivation of the energy principle, ideal inertia alone does not affect the threshold for instability. For the case of the $m = n = 1$ internal kink mode, which we know can be ideal-unstable, the addition of inertia enables removal of the singularity at $q = 1$, and for the ideal growth rate to be established.

The problem of finite inertia and resistivity is difficult. However, the problem becomes analytically tractable if the inertia and resistivity is treated carefully in the region where it has an impact, i.e. in a narrow region, known as the layer region around $q = m/n$. The local solution is then **matched using a layer variable** to the outer region where inertialess ideal MHD applies. The ideal MHD problem is generally quite easy to solve, while resistive problems are more delicate.

Inertia and ideal MHD

We start by defining the inertia $\delta K = -(\omega^2/2)\int \rho|\boldsymbol{\xi}|^2 d^3x$. For simplicity, we define the perpendicular inertia, defined in Eq. (3.26) so that we can eventually recover the perpendicular momentum equation of Eq. (1.8). We can bring in the effect of parallel inertia in an ad-hoc way later (exercises). Thus we have to the required order

$$\delta K_\perp = -\frac{\omega^2}{2}\int d^3x\,\rho\left(|\xi^r|^2 + |\xi^\theta|^2\right),$$

(where $\rho = m_i n_i$ is the mass density) and then substitute Eq. (3.28), i.e. $\xi_{\theta 0} = (i/m)\partial\xi_{r0}/\partial r$ so that $|\xi_\theta|^2 = (1/m^2)(d|r\xi_{r0}|/dr)^2$, giving to lowest order,

$$\delta K_\perp = -2\pi^2\omega^2\int_0^a \rho dr\, r\left[|\xi_{r0}|^2\left(1+\frac{1}{m^2}\right) + \frac{2r|\xi_0^r||(\xi_0^r)'|}{m^2} + \frac{r^2}{m^2}|(\xi_0^r)'|^2\right].$$

Defining the Alfvén toroidal frequency

$$\omega_A = \frac{v_A}{R_0} \quad \text{with} \quad v_A = \left(\frac{B_0^2}{\rho}\right)^{1/2} \tag{4.8}$$

then, $\delta K_\perp$ can be added to the first line of Eq. (3.33) for the ϵ^2 contribution to δW. to give the compact expression (dropping modulus notation) for internal modes

$$\delta K_\perp + \delta W_2 = \frac{2\pi^2 B_0^2}{R_0}\int_0^a dr\, r\left[\left(r\frac{d\xi_0^r}{dr}\right)^2 + \left(m^2-1\right)(\xi_0^r)^2\right]\left[\left(\frac{n}{m}-\frac{1}{q}\right)^2 + \frac{1}{m^2}\left(\frac{\gamma}{\omega_A}\right)^2\right]. \tag{4.9}$$

We use henceforth the notation for the total energy,

$$\delta H = \delta K_\perp + \delta W.$$

Inertia and ideal MHD: $m = 1$

The $m = 1$ mode is internal so we set a Dirichlet BC at $r = a$. The minimisation problem has already been seen to order ϵ^2, and at that order we can neglect inertia, so that the associated energy is:

$$\delta H_2 = \frac{2\pi^2 B_0^2}{R_0} \int_0^a dr\, r \left[\left(r \frac{d\xi^r_{G0}}{dr} \right)^2 \right] \left(n - \frac{1}{q} \right)^2$$

The solution to this, is a **global (G), or outer, solution** valid everywhere except very close to the rational surface, where inertia is important. It satisfies the Dirichlet BC at $r = a$:

$$\xi^r_{G0}(r) = \overline{\xi}_0 H(r - r_1)$$

where $H(r - r_1)$ is the Heaviside step function such that $H(r - r_1) = 1$ for $r < r_1$ and $H(r - r_1) = 0$ for $r > r_1$, where $q(r_1) = 1/n$, as seen from the Euler equation of Eq. (3.34). Notice if we substitute this solution back into δH_2 we obtain $\delta H_2 = 0$, so we are forced to go to the next order. The field line bending contribution now becomes order ϵ^4, still depending on the lowest order eigenfunction, but at this order the effects of inertia are important. We write,

$$\delta H_4 = \delta K_{4\perp}(\xi^r_0) + \delta W_{FLB}(\xi^r_0) + \delta W_{4G}(\xi^r_{G0}, \xi^r_{G1}, \xi^r_{G2})$$

where,

$$\delta K_{4\perp}(\xi^r_0) + \delta W_{FLB}(\xi^r_0) = \frac{2\pi^2 B_0^2}{R_0} \int_0^a dr\, r \left(r \frac{d\xi^r_0}{dr} \right)^2 \left[\left(n - \frac{1}{q} \right)^2 + \left(\frac{\gamma}{\omega_A} \right)^2 \right]$$

and $\delta W_{4G}(\xi^r_{G0}, \xi^r_{G1}, \xi^r_{G2})$ are the remaining order ϵ^4 potential energy contributions that are not sensitive to inertia corrections in ξ^r_0. **These are global, or outer contributions**. We note that near the rational surface $r = r_1$ (where $q(r_1) = 1/n$), we have that $\xi^r_0 - \xi^r_{G0} \sim \epsilon^0 \overline{\xi}_0$, so the leading order displacement must be corrected locally.

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The global (outer) energy at order ϵ^4 can be minimised independently of the other terms because ξ^r_{G0} is already fully defined (and ξ^r_{G1}, ξ^r_{G2} can be obtained by variation of δW_{G4} independently of inertia corrections). The result is Eq. (3.35), i.e.

$$\delta\hat{W}_{4G} = \left(1 - \frac{1}{n^2}\right)\delta\hat{W}^C_4 + \delta\hat{W}^T_4.$$

We now examine the inertia and field line bending contributions. To do this, we consider a **layer variable** $x = (r - r_1)/r_1$. The layer variable should be compared with the width δ over which the inertia is important. In particular, we have that,

$$\xi^r_0/\xi^r_{G0} \sim 1 \quad \text{for} \quad -\frac{\delta}{r_1} < x < \frac{\delta}{r_1} \tag{4.10}$$

$$\lim_{\frac{x}{\delta/r_1}\to-\infty} \xi^r_0(x) = \xi^r_{G0}(r = 0) = \overline{\xi}_0 \tag{4.11}$$

$$\lim_{\frac{x}{\delta/r_1}\to\infty} \xi^r_0(x) = \xi^r_{G0}(r = a) = 0 \tag{4.12}$$

We recall that,

$$\delta K_{4\perp}(\xi^r_0) + \delta W_{FLB}(\xi^r_0) = \frac{2\pi^2 B_0^2}{R_0}\int_0^a dr\, r\left(r\frac{d\xi^r_0}{dr}\right)^2\left[\left(n - \frac{1}{q}\right)^2 + \left(\frac{\gamma}{\omega_A}\right)^2\right].$$

Due to Eqs. (4.10) - (4.12) above, and the Heaviside step solution for ξ^r_{G0}, the integrand is significant only in the narrow region $-\delta/r_1 < x < \delta/r_1$. We can therefore write all variables in terms of x, expand the integrand around $x = 0$, and we are also free to take the limits of integration as $\pm\infty$. In these **inner layer contributions** we use the Taylor expansion (leading order term only) $n - 1/q(r) = ns_1x$, with $s_1 = (r/q)dq/dr$ at $x = 0$, describing the strongly spatially varying field line bending contribution. This gives:

Inertia: ideal internal kink problem

$$\delta K_{4\perp} + \delta W_{FLB} = 2\pi^2 R_0 B_0^2 \epsilon_1^2 n^2 s_1^2 \int_{-\infty}^{\infty} dx \left[x^2 + \left(\frac{\gamma}{n s_1 \omega_A} \right)^2 \right] \left(\frac{d\xi_0^r}{dx} \right)^2 . \tag{4.13}$$

Taking variations of this we obtain,

$$\frac{d}{dx} \left[\left\{ \left(\frac{\gamma}{n s_1 \omega_A} \right)^2 + x^2 \right\} \left(\frac{d\xi_0^r}{dx} \right) \right] = 0, \tag{4.14}$$

giving by integration

$$\frac{d\xi_0^r}{dx} = \frac{C}{x^2 + \left(\frac{\gamma}{n s_1 \omega_A} \right)^2} . \tag{4.15}$$

The constant of integration is obtained by integrating the above and using the boundary conditions Eq. (4.10) - (4.12) as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} dx \frac{d\xi_0^r}{dx} &= \int_{-\infty}^{\infty} dx \frac{C}{x^2 + \left(\frac{\gamma}{n s_1 \omega_A} \right)^2} \\ \xi_0^r(x \to \infty) - \xi_0^r(x \to -\infty) &= \pi C \left(\frac{\gamma}{n s_1 \omega_A} \right)^{-1} \\ -\bar{\xi}_0 &= \pi C \left(\frac{\gamma}{n s_1 \omega_A} \right)^{-1} . \end{aligned}$$

So that,

$$C = -\frac{\bar{\xi}_0}{\pi} \left(\frac{\gamma}{n s_1 \omega_A} \right), \quad \text{and thus} \quad \frac{d\xi_0^r}{dx} = -\frac{\bar{\xi}_0}{\pi} \left(\frac{\gamma}{n s_1 \omega_A} \right) \frac{1}{x^2 + \left(\frac{\gamma}{n s_1 \omega_A} \right)^2} . \tag{4.16}$$

Inertia: ideal internal kink problem

Substituting Eq. (4.16) into Eq. (4.13) yields

$$\delta K_{4\perp} + \delta W_{FLB} = 4R_0 B_0^2 \epsilon_1^2 n^2 s_1^2 \bar{\xi}_0^2 \left(\frac{\gamma}{n s_1 \omega_A}\right)^2 \int_0^\infty \frac{dx}{x^2 + \left(\frac{\gamma}{n s_1 \omega_A}\right)^2} = 2\pi R_0 B_0^2 \epsilon_1^2 n s_1 \bar{\xi}_0^2 \frac{\gamma}{\omega_A}. \quad (4.17)$$

We therefore obtain the dispersion, from $\delta H_4 = 0$:

$$\frac{\gamma}{\omega_A} = -\epsilon_1^2 \frac{\pi}{n s_1} \left[\left(1 - \frac{1}{n^2}\right) \delta \hat{W}_4^C + \frac{1}{n^2} \delta \hat{W}_4^T \right]. \quad (4.18)$$

where $\delta W = 2\pi^2 R_0 B_0^2 \bar{\xi}_0^2 \epsilon_1^4 \delta \hat{W}$

As mentioned earlier, it is seen that the relation

$$\gamma^2 \approx -\frac{\delta W_{min}}{K(\xi_{min})},$$

(where ξ_{min} is obtained from minimising δW alone) completely breaks down for resonant instabilities. The minimisation of δW alone correctly recovers the stability threshold, but any information regarding γ^2 has to be obtained through variation of the total energy $\delta K + \delta W$. This has been done above.

Furthermore, if one wishes to include the effect of the parallel displacement, as required for the full ideal MHD model, one simply replaces γ with $\gamma\sqrt{1 + 2q_s^2}$ in all the above dispersion relation (see exercises). Kinetic corrections create a strong modification to the Glasser - Greene - Johnson inertial enhancement. Graves, Hastie and Hopcraft were the first to do this [PPCF 2000], the inertial enhancement turns out to be identical to the collisonless zonal flow factor $(1 + 1.6q^2\epsilon^{-1/2})$, such that γ is replaced with $\gamma(1 + 1.6\epsilon^{-1/2})^{1/2}$ in the above dispersion relation.

Ideal layer width δ

To leading order, the perpendicular eigenfunction is incompressible also in the singular layer ($\nabla \cdot \xi_\perp = 0$), such that

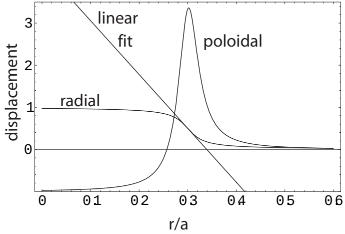
$$\xi_0^\theta = -\frac{i}{m}\left(\xi_0^r + r\frac{d\xi_0^r}{dr}\right)$$

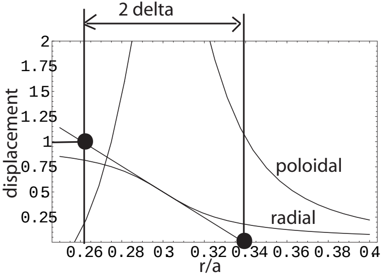
The figure plots ξ_0^r and ξ_0^θ with $\gamma/\omega_A = 0.025$, $r_1 = 0.3$ and $s_1 = 0.35$ all radial lengths are normalised to the plasma edge radius. Also plotted is a linear expansion of the radial eigenfunction around r_1, i.e.

$$\xi_0^r(fit) = (r - r_1)\frac{d\xi_0^r}{dr}\bigg|_{r_1} = \frac{1}{2} - \left(\frac{r - r_1}{r_1}\right)\frac{ns_1\omega_A}{\pi\gamma}$$

A characterisation of the width can be identified from the above linear fit, i.e.

$$\delta = \frac{\pi r_1 \gamma}{2ns_1\omega_A}.$$





Inertia and resistivity

Before considering the addition of plasma resistivity, let us first consider the variation of the energy defined by Eq. (4.9). Taking ω_A to be constant, or at least slowly varying with r compared to the other radially dependent terms in Eq. (4.9), we have,

$$\frac{d}{dr}\left[r^3\frac{d\xi_0^r}{dr}\left\{\left(\frac{n}{m}-\frac{1}{q}\right)^2+\frac{1}{m^2}\left(\frac{\gamma}{\omega_A}\right)^2\right\}\right]+(1-m^2)\left[\left(\frac{n}{m}-\frac{1}{q}\right)^2+\frac{1}{m^2}\left(\frac{\gamma}{\omega_A}\right)^2\right]r\xi_0^r=0, \tag{4.19}$$

which generalises Eq. (3.34) in order to account for perpendicular inertia.

The above compact expression conforms to the momentum equation, and normal-mode equation, subject to the ideal MHD model and in particular ideal Ohm's law. If we wish to include resistivity, it is convenient to re-arrange Eq. (4.19) to:

$$\left(\frac{\gamma}{\omega_A}\right)^2\left[\frac{d}{dr}(r^3(\xi_0^r)')+r\xi_0^r(1-m^2)\right]=-m^2\left\{\frac{d}{dr}\left[r^3(\xi_0^r)'\left(\frac{n}{m}-\frac{1}{q}\right)^2\right]+r\xi_0^r(1-m^2)\left(\frac{n}{m}-\frac{1}{q}\right)^2\right\} \tag{4.20}$$

This equation is clearly in the form

$$\partial^2\boldsymbol{\xi}/\partial t^2=\rho^{-1}(\boldsymbol{\delta J}\times\boldsymbol{B}+\boldsymbol{J}\times\boldsymbol{\delta B}-\boldsymbol{\nabla}\delta P)$$

where, at this order, δP does not enter. The RHS of Eq. (4.20) has been written in terms of the fluid displacement through the use of ideal Ohms law. We can remove the restriction of Ohms law simply by inverting its employment, and writing the RHS of Eq. (4.20) in terms of δB^r via Eq. (3.31)

$$\delta B_0^r=\frac{imB_0}{R_0}\left(\frac{n}{m}-\frac{1}{q}\right)\xi_0^r.$$

Inertia and resistivity

Giving

$$\left(\frac{\gamma}{\omega_A}\right)^2 \left[\frac{d}{dr}\left(r^3 \frac{d\xi_0^r}{dr}\right) + r\xi_0^r(1-m^2)\right] =$$
$$-im\frac{R_0}{B_0}\left\{r\left(\frac{n}{m}-\frac{1}{q}\right)\left(\frac{d}{dr}\left[r\frac{d}{dr}(r\,\delta B_0^r)\right] - m^2\delta B_0^r\right) + r^2\delta B_0^r \frac{R_0}{B_0}\frac{dJ_\phi}{dr}\right\} \quad (4.21)$$

where it has been convenient to use Eq. (3.32)) in order to write Eq. (4.21) in terms of dJ_ϕ/dr.

The power of Eq. (4.21) is evident; it is an eigenvalue equation that obeys the equation of motion at lowest order in ξ and δB, but is not subject to the assumption of ideal Ohm's law. i.e. we can now attempt to model the affect of resistivity on internal modes. Nevertheless, in order to examine resistive modes, we require an equation relating ξ^r to δB^r, this time subject to resistive Ohm's law. A suitable equation is obtained by taking the curl of linearised Ohm's law $\boldsymbol{\delta E} + \boldsymbol{\delta u} \times \boldsymbol{B} = \eta \boldsymbol{J}$ and applying Ampère's law on the right hand side,

$$-\frac{\partial \boldsymbol{\delta B}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{\delta u}_\perp \times \boldsymbol{B}) = \eta \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\delta B}).$$

Employing $\boldsymbol{\nabla} \times (\boldsymbol{\delta u}_\perp \times \boldsymbol{B}) = \boldsymbol{\delta u}_\perp(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\delta u}_\perp) + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{\delta u}_\perp - (\boldsymbol{\delta u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{B}$, and noting that $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ and, from Eq. (3.27) or (3.28) the leading order displacement, or velocity conforms to $\boldsymbol{\nabla}\cdot\boldsymbol{\delta u}_\perp = 0$, then choosing the radial component, and employing $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\delta B}) \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\delta B}) - \boldsymbol{\nabla}^2\boldsymbol{\delta B}$ then

$$\gamma\delta B^r + \boldsymbol{B}\cdot\boldsymbol{\nabla}\delta u^r \approx -\eta\boldsymbol{\nabla}^2\delta B^r$$

Inertia and resistivity

Now employing the definition of perturbed flux, given in Eq. (4.2), and the identities given in that section of these notes, it is straightforward to show that

$$\left(\frac{\gamma}{\omega_A}\right)^2 \left[\frac{d}{dr}\left(r^3 \frac{d\xi_0^r}{dr}\right) + r\xi_0^r(1-m^2)\right] = -m^2 \frac{R_0}{B_0}\left\{\left(\frac{n}{m}-\frac{1}{q}\right)\left(r\frac{d}{dr}\left[r\frac{d}{dr}\delta\psi\right] - m^2\delta\psi\right) + r\delta\psi\frac{R_0}{B_0}\frac{dJ_\phi}{dr}\right\} \quad \text{with} \tag{4.22}$$

$$\xi_0^r = \frac{R_0}{B_0 r}\left(\frac{n}{m}-\frac{1}{q}\right)^{-1}\left[\delta\psi - \frac{r\eta}{\gamma}\nabla^2\left(\frac{\delta\psi}{r}\right)\right], \tag{4.23}$$

Equations (4.22) and (4.23) form a complete eigenvalue equation for $\delta\psi$ that is quite rich in physics, at least if analytical solutions are sought. Fortunately, there is a means of making progress. Inertia, on the left hand side of Eq. (4.22) is only important very close to the rational surface, i.e where $q \approx m/n$. Assuming that growth rates are small, such that $\gamma/\omega_A \sim \epsilon^2$, then it is clear from Eqs. (4.22) and (4.23) that the inertia needs to be considered only when $|q - m/n| \sim \epsilon^2$ or less. In the outer region, where this is not the case, one simply solves Eq. (4.22) with the left hand side equal to zero,

$$r\frac{d}{dr}\left(r\frac{d\delta\psi}{dr}\right) + \left(\frac{R_0}{B_0}\right)\frac{rqm\delta\psi}{nq-m}\frac{dJ_\phi}{dr} - m^2\delta\psi = 0. \tag{4.24}$$

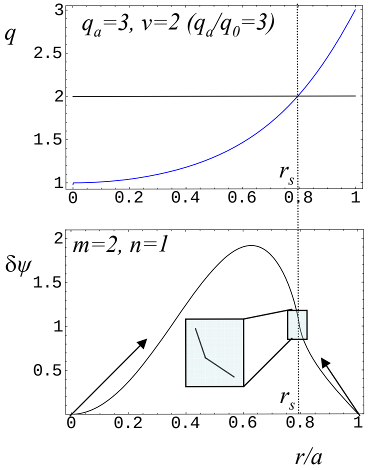
This equation is identical to Eq. (4.20) with LHS set to zero (setting inertia to zero in the ideal equation), or Eq. (3.34), or Eq. (4.7) for the displacement ξ_0^r, which can be solved for the flux via $\xi_0^r = (R_0/(B_0 r))\,(n/m - 1/q)^{-1}\,\delta\psi$ which is Eq. (4.23) in the ideal limit (appropriate for outer equations). In Eq. (4.24) for the flux, it is directly seen that ignoring the inertia in the region of the rational surface leads to an un-physical singularity, thus confirming that both Eq. (3.34) and Eq. (4.24) are only valid in the outer region.

Outer region

Equation (4.24) is solved for a particular m, n and $J_\phi(r)$ etc from the inner boundary $r = 0$, with BC's $\delta\psi(r=0) = 0$ and $\delta\psi'(r=0) = C_1$, and separately from an outer boundary, e.g. $r = a$ with BC's $\delta\psi(r=a) = 0$ and $\delta\psi'(r=a) = C_2$. In the first case, $\delta\psi$ is solved from $r = 0$ up to just within the rational surface $r_s - \delta r$, while in the second case, $\delta\psi$ is solved from $r = a$ up to just on the other side of the rational surface $r_s + \delta r$. We are permitted to assume an infinitely thin inner region, so that C_1 and C_2 are chosen such that $\delta\psi(r_s - \delta r) = \delta\psi(r_s + \delta r)$. But the two outer solutions approach r_s differently, such that $\delta\psi'(r_s - \delta r) \neq \delta\psi'(r_s + \delta r)$. The role of the inner region is to resolve this singularity, matching the outer region smoothly over a **resistive layer width** δ. Thus we define

$$\Delta' = \left.\frac{\delta\psi'}{\delta\psi}\right|_{r_s-\delta r}^{r_s+\delta r} \qquad \delta r \to 0 \qquad (4.25)$$

for the outer region. As will be seen, this is matched to the corresponding quantity evaluated in the inner region were we account for resistivity and inertia.



Constant-psi approximation and asymptotic matching

EPFL

The region close to the singular surface is often known as the inner region or layer region. In the layer region, $m - nq$ vanishes abruptly at the singular surface. The resistive layer width over which resistivity and inertia is important is about 2δ.

Resistive interchange and $m = 1$ **resistive internal kink modes** have the property that in the resistive layer $r_s \delta\psi'/\delta\psi \sim r_s/\delta \gg 1$.

But **tearing modes**, for which ξ_0^r is almost odd parity with respect to layer variable $x = (r - r_s)/r_s$ have the property,

$$\frac{r_s \delta\psi'}{\delta\psi} \sim 1. \tag{4.26}$$

That $\delta\psi$ varies very weakly across the layer will be exploited in the layer analysis to follow, in particular we will adopt the **constant-psi** approximation for tearing modes. Despite the weak variation in $\delta\psi$, the second derivative across the layer will not be insignificant. It is approximately,

$$\delta\psi'' \approx \frac{\delta\psi\, \Delta'_\delta}{2\delta} \quad \text{with} \quad \Delta'_\delta = \frac{\delta\psi'(x=\delta) - \delta\psi'(x=-\delta)}{\delta\psi(x=0)} \tag{4.27}$$

For matching of the inner region with the outer region we must calculate the asymptotic values of the appropriate variables through the inertial-resistive layer. Hence, in the layer region we evaluate,

$$\Delta' = \frac{1}{\delta\psi(x=0)} \left[\lim_{X\to\infty} \delta\psi'(X) - \lim_{X\to-\infty} \delta\psi'(X) \right]. \tag{4.28}$$

Constant-psi approximation and asymptotic matching

EPFL

While $\delta\psi$ varies slowly for tearing modes, the displacement always varies fast over the layer region, hence we expect $(\xi^r)' \sim \xi^r/\delta$, and $(\xi^r)'' \sim (\xi^r)'/\delta$, and hence

$$\frac{r_s^2(\xi^r)''}{\xi^r} \sim \frac{r_s^2}{\delta^2}.$$

Thus, from Eqs. (4.27) and (4.28)

$$\frac{r_s^2\delta\psi''}{\delta\psi} \sim \delta\,\Delta'\,\frac{r_s^2(\xi^r)''}{\xi^r}.$$

Tearing modes conforming to **the constant-psi approximation require that $\delta\psi$ varies in the layer much more weakly than ξ^r**. This can now be seen to require that

$$\delta\,\Delta' \ll 1. \tag{4.29}$$

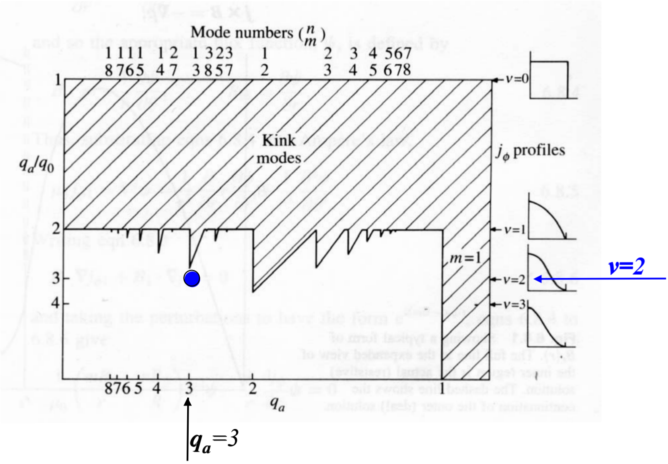
We note that by taking $\delta r \to 0$ in Eq. (4.25) for the calculation of Δ' in the outer region we evaluate the appropriate variables asymptotically in the ideal outer region, in preparation for matching with the inner region. Hence, for the outer region:

$$\Delta' = \frac{1}{\delta\psi(r_s)}\lim_{\delta r\to 0}[\delta\psi'(r_s+\delta r) - \delta\psi'(r_s-\delta r)] = \lim_{\delta r\to 0}\left.\frac{\delta\psi'}{\delta\psi}\right|_{r_s-\delta r}^{r_s+\delta r}, \tag{4.30}$$

where as mentioned earlier, $\delta\psi$ is obtained from the solution of Eq. (4.24) with appropriate boundary conditions applied. This outer calculation of Δ' will be matched to the inner layer Δ' of Eq. (4.28). This is required because the inner layer calculation provides a relationship between the growth rate and Δ', but in the layer the value of Δ' is an unknown (it must be matched from the determinable solution of the outer equations). Typically found that $r_s\Delta' \sim 1$, so that Eq. (4.29) is met for $\delta/r_s \ll 1$.

Current driven tearing modes

Recall that ideal internal modes are stable at order ϵ^2. Pressure gradient effects and toroidal effects are absent at order ϵ^2, so only the drive from current gradient exists. We found that external kink modes could be driven unstable with a certain choice of current profile, or q-profile. This domain of instability for a given equilibrium current profile can be extended by inclusion or resistive effects.



This case is stable to Ideal External Kink modes (even m=3, n=1 mode).
We can investigate stability of Internal Resistive Tearing Mode (n=1, m=2). It is UNSTABLE .