Lecture 3

Theory of linear ideal MHD stability

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Linear MHD equations

Allow all the MHD variables (electromagnetic fields and fluid variables) to comprise the sum of an equilibrium component and a perturbation which is small in comparison:

$$\boldsymbol{Q} \rightarrow \boldsymbol{Q} + \boldsymbol{\delta Q}.$$

For example, employing the normal mode approach, the perturbations acquire the form $\boldsymbol{\delta Q}(\boldsymbol{x}, t) = \boldsymbol{\delta Q}(\boldsymbol{x}) \exp(-i\omega t)$, representing disturbances that have always existed and do not require initial conditions.

In MHD it is usually assumed that the equilibrium velocity $\boldsymbol{u}$ is negligible. This approximation is valid providing the centrifugal forces associated with a circulating fluid is small. It is convenient to define the perturbed fluid velocity in terms of the fluid displacement $\boldsymbol{\xi}$ such that

$$\boldsymbol{\delta u} = \frac{\partial \boldsymbol{\xi}}{\partial t}.$$

Here we have used the fact that there is no initial displacement at $t = 0$. Let us start by assuming the standard ideal MHD momentum equation, and then linearise in perturbations:

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \boldsymbol{\delta F}, \tag{3.1}$$

where

$$\boldsymbol{\delta F} = \boldsymbol{\delta J} \times \boldsymbol{B} + \boldsymbol{J} \times \boldsymbol{\delta B} - \boldsymbol{\nabla} \delta P.$$

Again, we have assumed that the equilibrium $\boldsymbol{u}$ is zero (i.e. no equilibrium plasma flows or momenta). The perturbed pressure can be written in terms of the fluid displacement as follows,

$$\delta P = -\gamma P \boldsymbol{\nabla} \cdot \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P. \tag{3.2}$$

Linear force operator

This was obtained by taking the adiabatic equation of state $(d/dt)(P\rho^{-\gamma}) = 0$ and linearising to give:

$$\frac{\partial \delta P}{\partial t} + \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \boldsymbol{\nabla} P = \frac{P\gamma}{\rho}\left(\frac{\partial \delta\rho}{\partial t} + \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \boldsymbol{\nabla}\rho\right),$$

which must be combined with an equation for the rate of change of the perturbed mass density, i.e. the mass density equation $d\rho/dt + \rho\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, so that $\partial\delta\rho/\partial t + \boldsymbol{\nabla}\cdot(\rho\partial\boldsymbol{\xi}/\partial t) = 0$, to give the result (by integration).

We see that the perturbed pressure contains the adiabatic effect of $-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P$ which is caused by the displacement of the fluid orientated in the direction of the equilibrium pressure gradient. The equilibrium pressure gradient is perpendicular to the equilibrium field lines, hence $-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}P = -\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\nabla}P$. Meanwhile, $-\gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ is the effect of compressibility, and the effect necessarily involves $\boldsymbol{\xi}_{\parallel}$. Incompressibility requires $d\delta\rho/dt = 0$ which in turn requires

$$\boldsymbol{\nabla}\cdot\boldsymbol{\delta u} = \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0.$$

The linearised magnetic field $\boldsymbol{\delta B}$ can be calculated by combining Faraday's Law with Ohm's law, which integrates to give

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}), \tag{3.3}$$

and consequently, from Amperes law, the perturbed current is

$$\boldsymbol{\delta J} = \boldsymbol{\nabla}\times\boldsymbol{\delta B} = \boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B})]$$

thus giving the linearised perturbed force

$$\boldsymbol{\delta F}(\boldsymbol{\xi}) = \boldsymbol{\nabla}(\gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}P) + [(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}))\times\boldsymbol{B} + (\boldsymbol{\nabla}\times\boldsymbol{B})\times(\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}))]. \tag{3.4}$$

Approaches for solving MHD equations

The equation of motion Eq. (3.1) and Eq. (3.4) is now entirely in terms of the fluid displacement vector $\boldsymbol{\xi}$ and the equilibrium magnetic field. There are at least four ways of deploying Eq. (3.1) and Eq. (3.4) in order to analyse MHD stability, and these are listed in order of their complexity (from most difficult to least difficult)

1. The Initial Value Problem
2. The Normal Mode Method
3. Variation of the total Energy
4. Energy Principle

The Initial Value Problem

Equations (3.1) and (3.4) are solved as an initial value problem. The complexity of the MHD force operator usually means that this approach must be tackled computationally. Whilst it yields a detailed description of the evolution of a perturbation, it is often the case that this information is in excess of what is required to characterise an instability. Hence the method lacks the overall power of the other techniques. Nevertheless an advantage is that nothing is assumed about the form of the time dependence of the perturbations (i.e. not necessarily normal modes), so that the method provides a framework consistent with a non-linear treatment.

Conservation of energy

Before the normal mode, variational energy, and energy principle methods are considered, it is useful to demonstrate that linear MHD perturbations conform to the conservation of energy. The proof requires the definition of important and compact energy related quantities.

First, apply $\int d^3x\, \dot{\boldsymbol{\xi}}^* \cdot$ to the RHS and LHS of Eq. (3.1), where the volume integral is over all the plasma volume so that

$$\int d^3x\, \rho\, \dot{\boldsymbol{\xi}}^* \cdot \ddot{\boldsymbol{\xi}} = \int d^3x\, \dot{\boldsymbol{\xi}}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi}), \tag{3.5}$$

and since $\partial/\partial t(\dot{\boldsymbol{\xi}}^* \cdot \dot{\boldsymbol{\xi}}) = \dot{\boldsymbol{\xi}}^* \cdot \ddot{\boldsymbol{\xi}} + \ddot{\boldsymbol{\xi}}^* \cdot \dot{\boldsymbol{\xi}} = 2\dot{\boldsymbol{\xi}}^* \cdot \ddot{\boldsymbol{\xi}}$, then,

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int d^3x\, \rho\, \left|\dot{\boldsymbol{\xi}}\right|^2\right) = \int d^3x\, \dot{\boldsymbol{\xi}}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi}). \tag{3.6}$$

This is almost in the form of an energy conservation equation. A little bit more work is required for this. Now,

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int d^3x \boldsymbol{\xi}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi})\right] = \frac{1}{2}\int d^3x\, \dot{\boldsymbol{\xi}}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi}) + \frac{1}{2}\int dx^3\, \boldsymbol{\xi}^* \cdot \boldsymbol{\delta F}(\dot{\boldsymbol{\xi}})$$

where we used $\boldsymbol{\delta}\dot{\boldsymbol{F}}(\boldsymbol{\xi}) = \boldsymbol{\delta F}(\dot{\boldsymbol{\xi}})$ A VERY IMPORTANT PROPERTY IS NOW REQUIRED. We will not prove it (see Freidberg for details). We require the self adjointness property

$$\int d^3x\, \boldsymbol{\xi_1} \cdot \boldsymbol{\delta F}(\boldsymbol{\xi_2}) = \int d^3x\, \boldsymbol{\xi_2} \cdot \boldsymbol{\delta F}(\boldsymbol{\xi_1}), \tag{3.7}$$

Now, since $\boldsymbol{\delta F}(\boldsymbol{\xi})$ is linear in $\boldsymbol{\xi}$, and since the time dependence in $\boldsymbol{\delta F}$ resides entirely in $\boldsymbol{\xi}$ then $\boldsymbol{\delta}\dot{\boldsymbol{F}}(\boldsymbol{\xi}) = \boldsymbol{\delta F}(\dot{\boldsymbol{\xi}})$. Thus together with the self-adjointness property we have,

$$\frac{1}{2}\int dx^3\, \boldsymbol{\xi}^* \cdot \boldsymbol{\delta F}(\dot{\boldsymbol{\xi}}) = \frac{1}{2}\int dx^3\, \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\delta F}(\boldsymbol{\xi}^*) \quad \text{and therefore} \quad \int d^3x\, \dot{\boldsymbol{\xi}}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi}) = \frac{1}{2}\frac{\partial}{\partial t}\left[\int d^3x\, \boldsymbol{\xi}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi})\right].$$

Normal mode method

Therefore, Eq. (3.6) can be written as the following energy conservation equation:

$$\frac{\partial}{\partial t}\left[\delta K + \delta W\right] = 0 \tag{3.8}$$

where

$$\delta K = \frac{1}{2}\int d^3x\,\rho\,\left|\dot{\boldsymbol{\xi}}\right|^2 \quad \text{and} \quad \delta W = -\frac{1}{2}\int d^3x\,\boldsymbol{\xi}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi}). \tag{3.9}$$

Here, δK is the kinetic energy, and δW is the potential energy. One then clearly has

$$\delta K + \delta W = \text{constant}.$$

Conservation of Energy

For the study of spontaneously occurring instabilities, one assumes that the perturbations are normal modes of the form

$$\boldsymbol{\xi} \sim \exp(-i\omega t) \quad (\text{and therefore } \boldsymbol{\xi}^* \sim \exp(i\omega t)).$$

With the normal mode method one then solves (from Eq. (3.1))

$$-\omega^2\rho\boldsymbol{\xi} = \boldsymbol{\delta F}(\boldsymbol{\xi}), \tag{3.10}$$

for the eigenvalue ω and each component of the eigenvector, where $\boldsymbol{\delta F}$ is given by (3.4).

Normal mode method

In general, the eigenvalues can only be found by obtaining the solutions of three coupled partial differential equations. Nevertheless, if the system contains a degree of symmetry (slab, cylinder, axisymmetric torus) the complexity is reduced.

Before embarking on applications, another very important property is required. Substituting the normal mode assumption into Eq. (3.5) one obtains

$$i\omega^3 \int d^3x\, \rho\, |\boldsymbol{\xi}|^2 = i\omega \int d^3x\, \boldsymbol{\xi}^* \cdot \boldsymbol{\delta F}(\boldsymbol{\xi})$$

and this directly ensures that for normal modes, the constant in the energy conservation equation (Eq. 3.8) is zero

$$\delta K + \delta W = 0 \tag{3.11}$$

so that

$$\omega^2 = \frac{\delta W}{K} \quad \text{with} \quad K = \frac{1}{2} \int d^3x\, \rho\, |\boldsymbol{\xi}|^2 \tag{3.12}$$

where

$$\delta K = -\omega^2 K$$

It is therefore clear that ω^2 is real. Normal modes therefore conform to either:

1. Two stationary modes, with one growing and the other decaying (i.e. $\omega^2 < 0$), so that $\xi \sim \exp(\pm i|\omega|t)$
2. Two modes neither growing or decaying but both propagate with equal speeds and in the opposite sense (i.e. $\omega^2 > 0$), so that $\xi \sim \exp(\pm|\omega|t)$

Thus, $\omega = 0$ marks the transition between purely growing (or decaying) modes, and purely propagating modes. Finally we note that Eqs. (3.11) and (3.12) are very important for the variational energy method and energy principle method, as we will see later.

There are an infinite number of oscillations, or modes that conform to Eq. (3.11) or equivalently Eq. (3.12). Conceptually, it is clear that we are primarily interested in the most unstable modes, thus giving the largest growth rates. In the following, we will show that the maxima or minima of

$$\frac{\delta W}{K},$$

with respect to variation over $\boldsymbol{\xi}$, conforms to the equation of motion (i.e. the normal mode equation Eq. (3.10)), and moreover, since $\omega^2 = \delta W/K$, the minima (or maxima in $-\omega^2$) clearly identifies the most unstable mode.

We start by perturbing the "eigenvector" and "eigenvalue":

$$\omega^2 \to \omega^2 + \delta\omega^2, \quad \boldsymbol{\xi} \to \boldsymbol{\xi} + \boldsymbol{\delta\xi}, \quad \boldsymbol{\xi}^* \to \boldsymbol{\xi}^* + \boldsymbol{\delta\xi}^*,$$

so that Eq. (3.12) becomes

$$\omega^2 + \delta\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\delta\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\xi}^*, \boldsymbol{\delta\xi}) + \delta W(\boldsymbol{\delta\xi}^*, \boldsymbol{\delta\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\delta\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\xi}^*, \boldsymbol{\delta\xi}) + K(\boldsymbol{\delta\xi}^*, \boldsymbol{\delta\xi})}$$

Energy method for analysing stability

There are an infinite number of oscillations, or modes that conform to Eq. (3.11) or equivalently Eq. (3.12). Conceptually, it is clear that we are primarily interested in the most unstable modes, thus giving the largest growth rates. In the following, we will show that the maxima or minima of

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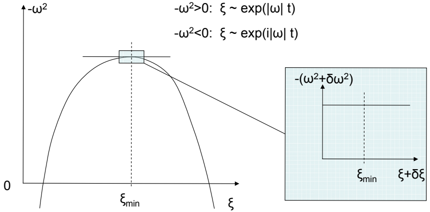


We note that we want to find the extrema of the above wrt $\boldsymbol{\xi}$, which clearly requires $\delta\omega^2 = 0$.

$$\omega^2 = \left.\frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\delta\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\xi}^*, \boldsymbol{\delta\xi}) + \delta W(\boldsymbol{\delta\xi}^*, \boldsymbol{\delta\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\delta\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\xi}^*, \boldsymbol{\delta\xi}) + K(\boldsymbol{\delta\xi}^*, \boldsymbol{\delta\xi})}\right|_{\boldsymbol{\delta\xi}\rightarrow 0, \boldsymbol{\xi}\rightarrow\boldsymbol{\xi}_{\rm min}}$$

Energy method for analysing stability

Retaining $\delta\omega^2$ for now, and Taylor expanding the denominator, we have

$$\omega^2+\delta\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})+\delta W(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})+\delta W(\boldsymbol{\delta\xi}^*,\boldsymbol{\delta\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}\left[1-\frac{K(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}-\frac{K(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}-O(\frac{K(\delta^2)}{K})\right]$$

Continuing by linearising in small δ we have

$$\omega^2+\delta\omega^2 = \frac{\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}+\frac{\delta W(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}-\frac{\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}\left[\frac{K(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}+\frac{K(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})}{K(\boldsymbol{\xi}^*,\boldsymbol{\xi})}\right]$$

Now substituting $\omega^2=\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})/K(\boldsymbol{\xi}^*,\boldsymbol{\xi})$ we have,

$$\delta\omega^2 K(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = \delta W(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})-\omega^2\left(K(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})+K(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})\right).$$

or, since $\delta K=-\omega^2 K$,

$$-\frac{\delta\omega^2}{\omega^2}\delta K(\boldsymbol{\xi}^*,\boldsymbol{\xi}) = \delta W(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi})+\delta K(\boldsymbol{\delta\xi}^*,\boldsymbol{\xi})+\delta K(\boldsymbol{\xi}^*,\boldsymbol{\delta\xi}). \qquad (3.13)$$

Thus, the extremum in ω^2, for which $\delta\omega^2=0$, conforms to an extremum (or stationary point) of

$$\delta K(\omega^2,\boldsymbol{\xi}^*,\boldsymbol{\xi})+\delta W(\boldsymbol{\xi}^*,\boldsymbol{\xi})$$

over variation with respect to $\boldsymbol{\xi}$ for constant ω. We now have to demonstrate that the stationary point of $\delta K+\delta W$ conforms to the normal mode equation.

Energy method for analysing stability

Examining Eq. (3.13) we have:
$0 = \int d^3x \left\{ \boldsymbol{\delta\xi}^* \cdot \delta F(\boldsymbol{\xi}) + \boldsymbol{\xi}^* \cdot \delta F(\boldsymbol{\delta\xi}) + \omega^2 \rho \boldsymbol{\delta\xi}^* \cdot \boldsymbol{\xi} + \omega^2 \rho \boldsymbol{\xi}^* \cdot \boldsymbol{\delta\xi} \right\}.$
Now, applying the self adjoint property of Eq. (3.7),

$$\int d^3x\, \boldsymbol{\xi}^* \cdot \delta F(\boldsymbol{\delta\xi}) = \int d^3x\, \boldsymbol{\delta\xi} \cdot \delta F(\boldsymbol{\xi}^*)$$

we have

$$0 = \int d^3x\, \left\{ \boldsymbol{\delta\xi}^* \cdot \left[\omega^2 \rho \boldsymbol{\xi} + \delta F(\boldsymbol{\xi}) \right] + \boldsymbol{\delta\xi} \cdot \left[\omega^2 \rho \boldsymbol{\xi}^* + \delta F(\boldsymbol{\xi}^*) \right] \right\},$$

Because $\boldsymbol{\delta\xi}^*$ is arbitrary, it means that the stationary point of $\delta K + \delta W$ over a variation wrt $\boldsymbol{\xi}$ for constant ω^2, reproduces the equation of motion $\omega^2 \rho \boldsymbol{\xi} + \delta F(\boldsymbol{\xi}) = 0$.

In much of the remaining parts of this course, we will obtain the linear growth rate by minimisation of $\delta K + \delta W$ via the Euler-Lagrange equations. Typically, much of the minimisation will be done algebraically, leaving the total energy to be minimised in terms of only the radial component of the displacement,

$$\delta K + \delta W = \int_0^{\text{edge}} dr\, I(\omega^2, r, \xi_r, \xi_r')$$

for some I to be identified, and $\xi_r' = d\xi_r/dr$. The extremum of $\delta K + \delta W$ with respect to variation over ξ_r for constant ω^2 is given by the Euler-Lagrange equation:

$$\frac{\partial I}{\partial \xi_r} - \frac{d}{dr} \left\{ \frac{\partial I}{\partial \xi_r'} \right\} = 0. \qquad (3.14)$$

Energy principle

The primary interest is to know simply whether a particular equilibrium is linearly unstable to a particular type of instability, or not. The exact eigenvalue and eigenvector might be surplus to requirements. If this is the case, the energy principle is the most efficient way to proceed with stability (or instability) analysis. From Eq. (3.12), and noting that $K(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$ is positive definite, we have

$$\mathrm{sign}\{\omega^2\} = \mathrm{sign}\{\delta W\}$$

where we note that the RHS depends only on $\boldsymbol{\xi}$, and is independent of ω^2. One can then analyse whether an equilibrium has the 'potential' to be unstable by applying all physically allowable trial functions in δW to see if any of them generate $\mathrm{sign}\{\delta W\} = -1$, i.e. instability. Mathematically this is equivalent to finding the minimum of δW over variation in $\boldsymbol{\xi}$, and to see if this minimum δW is negative.

We note that the extremum of δW does not in general conform to the normal mode equation, since the inertia δK has been neglected. Nevertheless, close to marginal stability, it *might* be expected that the ξ_{min}, which satisfies $\delta W(\xi_{min}) = \delta W_{min}$, could be employed to give

$$\omega^2 \approx \frac{\delta W_{min}}{K(\xi_{min})}?$$

However, the above relation completely breaks down for some modes, such as the internal kink mode. This apparent anomaly remains true even as we approach the stability boundary, for which $\omega^2 \to 0$. For the internal kink mode, we will see that minimisation of the total energy reveals $K \propto 1/\omega$, so that $\omega \propto \delta W$.

The minimisation of δW does correctly recover the stability threshold (which will identify conditions for instability such as threshold pressure gradient - this is the concept of the Energy Principle), but any information regarding ω^2 has to be obtained through variation of the total energy $\delta K + \delta W$.

Convenient form for δW

First, insert Eq. (3.4), using Eq. (3.3) into Eq. (3.9) and integrate by parts (use divergence theorem) to give,

$$\delta W = \frac{1}{2}\int_P d^3x \left\{|\boldsymbol{\delta B}|^2 - \boldsymbol{\xi}^* \cdot [\boldsymbol{J}\times\boldsymbol{\delta B} + \boldsymbol{\nabla}(\boldsymbol{\xi}\cdot\nabla P)] + \gamma P|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2\right\} - \frac{1}{2}\int dS\,(\boldsymbol{n}\cdot\boldsymbol{\xi}^*)\,(\gamma P\boldsymbol{\nabla}\cdot\boldsymbol{\xi} - \boldsymbol{B}\cdot\boldsymbol{\delta B})$$

where $\boldsymbol{n}$ is a vector pointing normal to the plasma surface S, and dS is a surface element, covering the plasma-vacuum interface. Here subscript P denotes integration over the plasma volume.

It can be shown that only the perpendicular component of $\boldsymbol{\xi}^*$, i.e. $\boldsymbol{\xi}^*_\perp$, survives in $\boldsymbol{\xi}^* \cdot [\boldsymbol{J}\times\boldsymbol{\delta B} + \boldsymbol{\nabla}(\boldsymbol{\xi}\cdot\nabla P)]$ on integration (see Eqs. (4.24) - (4.27) of White "Theory of Toroidally Confined Plasmas, 2nd Edition"). Further simplifications are made by integration by parts. The surface integral can be converted to a volume integral over the vacuum region (denoted V), to obtain

$$\delta W = \delta W_P + \delta W_V$$

$$\text{where}\quad \delta W_P = \frac{1}{2}\int_P d^3x \left\{|\boldsymbol{\delta B}|^2 - \boldsymbol{\xi}^*_\perp\cdot(\boldsymbol{J}\times\boldsymbol{\delta B}) + (\boldsymbol{\xi}_\perp\cdot\nabla P)\boldsymbol{\nabla}\cdot\boldsymbol{\xi}^*_\perp + \gamma P|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2\right\}, \quad (3.15)$$

$$\text{and}\quad \delta W_V = \frac{1}{2}\int_V d^3x \left|\boldsymbol{\delta\hat{B}}\right|^2 \quad (3.16)$$

where $\boldsymbol{\delta\hat{B}}$ is the perturbed magnetic field in the vacuum region, which is coupled to plasma displacement by solving $\boldsymbol{\nabla}\cdot\boldsymbol{\delta\hat{B}} = \boldsymbol{\nabla}\times\boldsymbol{\delta\hat{B}} = 0$, subject to the conditions ($\boldsymbol{n}$ is normal to a conducting wall),

$$\boldsymbol{n}\cdot\boldsymbol{\delta\hat{B}}|_b = 0 \quad\text{and}\quad \boldsymbol{n}\cdot\boldsymbol{\delta\hat{B}}|_a = \boldsymbol{n}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\xi}_\perp\times\boldsymbol{B})|_a \quad (3.17)$$

where b is the conducting wall 'radius' (surface position) and a is the 'radius' (surface position) of the plasma-vacuum interface, so that $a < b$.

Convenient form for δW

This nearly completes the derivation of the intuitive form of δW. What remains is to separate $|\delta B|$ and $\boldsymbol{J}$ into perpendicular and parallel components. Use is made of (see exercises)

$$\boldsymbol{J}_\perp = \frac{B \times \boldsymbol{\nabla} P}{B^2} \quad \text{and} \quad \delta B_\parallel = -B\left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}\right) + \frac{\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} P}{B} \tag{3.18}$$

where the curvature $\kappa = (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{b}$, (where $\boldsymbol{b} = \boldsymbol{B}/B$) to give for Eq. (3.15)

$$\delta W_P(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \delta W_\perp(\boldsymbol{\xi}_\perp, \boldsymbol{\xi}_\perp^*) + \overline{\delta W}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) \quad \text{where,} \tag{3.19}$$

$$\delta W_\perp(\boldsymbol{\xi}_\perp, \boldsymbol{\xi}_\perp^*) = \frac{1}{2}\int_P d^3x \left[|\delta B_\perp|^2 + B^2\left|\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}\right|^2 - 2(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} P)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp^*) - J_\parallel(\boldsymbol{\xi}_\perp^* \times \boldsymbol{b}) \cdot \boldsymbol{\delta B}_\perp\right] \tag{3.20}$$

$$\overline{\delta W}(\boldsymbol{\xi}, \boldsymbol{\xi}^*) = \frac{1}{2}\int_P d^3x\, \gamma P(\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2, \tag{3.21}$$

where $J_\parallel$ is the parallel equilibrium current density.

Since $\boldsymbol{\delta B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \boldsymbol{B})$ then it is seen that $\delta W_\perp$ is independent of $\boldsymbol{\xi}_\parallel$. Each of the terms in Eq. (3.20) has a simple physical interpretation.

- The first term in $\delta W_\perp$ is always stabilising and is the magnetic energy in the Alfvén wave associated with field line bending.
- The second term is also stabilising and corresponds to the energy necessary to compress the magnetic field and describes the major potential energy contribution to magnetosonic waves.
- The third term, proportional to the pressure gradient, is the potential energy drive for the ballooning and interchange instabilities. It is destabilising if $\boldsymbol{\nabla} P$ and $\boldsymbol{\kappa}$ are parallel to one another (unfavourable curvature).
- The fourth term is the free energy arising from the current and is responsible for kink instabilities. Finally, $\overline{\delta W}$ is the energy required to compress the plasma. It is stabilising.

Incompressibility

Before using the energy method it is essential to attempt a reduction in the complexity of the analysis. Here we demonstrate that under certain conditions, the dimensions of the system can be reduced from three to two.

Equation (3.20) shows that $\delta W_\perp$ is independent of $\boldsymbol{\xi}_\parallel$. This means that δW can be minimised with respect to $\boldsymbol{\xi}_\parallel$ by only considering $\overline{\delta W}$. It is clear that $\overline{\delta W}$ is positive definite and minimises to zero when $\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$. This requires that the parallel flow satisfies

$$B \frac{\partial}{\partial l} \left(\frac{\boldsymbol{\xi}_\parallel}{B} \right) = -\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp . \tag{3.22}$$

Near marginal stability, from the point of view of obtaining the eigenfunction, variation of δW is equivalent to the variation of $\delta K + \delta W$. Hence near marginal stability (weak growth rate), instabilities are nearly incompressible, and the parallel displacement will be approximately given by Eq. (3.22). Thus near marginal stability, δW can be written in terms of $\boldsymbol{\xi}_\perp$ alone. Note though that the total energy depends on the parallel displacement via $\delta K(\boldsymbol{\xi}_\perp, \boldsymbol{\xi}_\parallel)$. As shown in the exercises, inclusion of $\boldsymbol{\xi}_\parallel$ in δK renormalises the growth rate compared to the incompressible MHD model codes (e.g. TERPISCHORE) that ignore it. The renormalisation is $\gamma^2 \rightarrow (1 + 2q^2)\gamma^2$ (as seen in exercise series 4).

One should note that the parallel displacement does not satisfy Eq. (3.22) far from the stability boundary, i.e. normal modes are not incompressible when their rate of growth is large. Where the effects of compressibility are expected to be important, the adiabatic model, which generates $\overline{\delta W}(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$, is lacking in rigour, and a more sophisticated model is required to account for the parallel dynamics.

For most of this course, the incompressible MHD model is investigated (i.e. we take variations of $\delta K_\perp + \delta W_\perp + \delta W_V$). At the end of the course, kinetic effects are included in order to account for 'kinetic compressibility,' i.e. the kinetic analogue of $\overline{\delta W}(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$. We now briefly comment on a few fundamental concepts of kinetic MHD.

Comment: alternative kinetic-MHD closure

The aim is to avoid the adiabatic equation of state for the energy equation $\frac{d}{dt}\left(P\rho^{-\gamma}\right)$, which yields Eq. (3.2), i.e $\delta P = -\gamma P \boldsymbol{\nabla}\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\boldsymbol{\nabla} P$, and replace this with a kinetic closure which ultimately yields

$$\underline{\underline{\delta P}} = \underline{\underline{I}}(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla} P) + \underline{\underline{\delta P}}_k \tag{3.23}$$

where the pressure tensor conforms to the diagonal tensor defined in Eq. (1.7). Also $\underline{\underline{\delta P}}_k$ will be defined later, but suffice to say at present that $\underline{\underline{\delta P}}_k$ depends explicitly on $\boldsymbol{\xi}_\perp$ and ω.

Rather than employing the full MHD equations, we choose the (more valid) perpendicular MHD equations defined by Eqs. (1.8), derived earlier. Ignoring equilibrium plasma flows, one then has the perpendicular normal mode equation,

$$-\omega^2\rho\boldsymbol{\xi}_\perp = \boldsymbol{\delta F}_\perp(\boldsymbol{\xi}_\perp), \tag{3.24}$$

with

$$\begin{aligned}\boldsymbol{\delta F}_\perp(\boldsymbol{\xi}_\perp) \quad = \quad & [\boldsymbol{\nabla} - \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})](\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla} P) - [\boldsymbol{\nabla} - \boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})]\delta P_{k\perp} - (\delta P_{k\parallel} - \delta P_{k\perp})\boldsymbol{\kappa} + \\ & [(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times(\boldsymbol{\xi}_\perp\times\boldsymbol{B}))\times\boldsymbol{B} + (\boldsymbol{\nabla}\times\boldsymbol{B})\times(\boldsymbol{\nabla}\times(\boldsymbol{\xi}_\perp\times\boldsymbol{B}))]_\perp .\end{aligned} \tag{3.25}$$

We now follow the energy method of ideal MHD, but this time, form the dot product of Eq. (3.24) with $\boldsymbol{\xi}_\perp^*$ and integrate over all space, giving

$$\delta K_\perp + \delta W_\perp + \delta W_V + \delta W_k = 0 \tag{3.26}$$

where

$$\delta K_\perp = -\omega^2\frac{1}{2}\int d^3x\,\rho\,|\boldsymbol{\xi}_\perp|^2 \quad \text{and} \quad \delta W_k = -\frac{1}{2}\int d^3x\,\left[\delta P_{k\perp}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*) - (\delta P_{k\parallel} - \delta P_{k\perp})\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa}\right]$$

From inspection, it is clear that $\delta W_\perp$ is simply that defined in Eq. (3.20), and δW_V is simply that defined in Eq. (3.16).

Comment: alternative kinetic-MHD closure

This is obvious because the sum of the δW contributions has to be identical if we set $\gamma \to 0$ and $\boldsymbol{\xi} \to \boldsymbol{\xi}_\perp$ in the MHD model, and $\delta P_k = 0$ in the kinetic model. Hence, δW_k has replaced the compressibility term, $\overline{\delta W}$, of the MHD model. One other subtle difference is that $\delta K_\perp$ has replaced δK. This modification to the inertia is currently being explored by a number of researchers, but it is out of the scope of this course (I can supply references for those interested)

We note that Eq. (3.26) forms a dispersion relation, and if the force operator associated with δW_k is self-adjoint, one is permitted to apply variation to Eq. (3.26), and thereby recover the perpendicular equation of motion Eq. (3.24). Thus we have been able to avoid the MHD parallel equation of motion. Parallel dynamics will be controlled by the drift kinetic equation. In principle the latter obtains a kinetic equation analogous to Eq. (3.22), so that the parallel displacement is not required explicitly.

The generalised dispersion relation of Eq. (3.26) is extremely powerful. It is capable of including the physics of wave-particle interaction, e.g. Landau damping. Modes frequently investigated include the internal kink mode, including the interaction of fast ions on sawteeth and fishbones. Also investigated are kinetic ballooning modes, interchange modes, infernal modes, resistive wall modes, and many others.

Inverse aspect ratio expansion of stability equations

We will consider stability to large scale modes (long wavelengths). We will expand all quantities in inverse aspect ratio. At lowest order in ϵ, the results are identical in a torus and a cylinder. The derivations are a little different from those typically found in textbooks, which often start by assuming cylindrical geometry.

In an axisymmetric torus, a perturbation may be described by a Fourier decomposition in poloidal harmonics as

$$\boldsymbol{\xi}(r,\theta,\phi) = \sum_m \boldsymbol{\xi}^{(m)}(r) e^{i(n\phi - m\theta - \omega t)}$$

by virtue of the periodicity in the poloidal and toroidal directions. Since the equilibrium magnetic field strength $B \approx B_0(1-\epsilon\cos\theta)$ is a function of θ, coupling in the different poloidal harmonics arises in $\boldsymbol{\xi}$. This coupling will not appear in the present lecture since only lowest order effects are considered at present, and as a consequence, we do not need to be so careful about the definition of θ (we often assume straight field line coordinates, more details in the exercise series).

We will minimise δW with respect to $\boldsymbol{\xi}$. We permit ourselves to introduce the inertia δK later, should growth rates be required, and we can order δK as required (depends on closeness to stability threshold). To aid the analysis the eigenvector is expanded as follows:

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2,$$

where the subscript denotes inverse aspect ratio $\epsilon = r/R_0$ ordering of each term, e.g. $\boldsymbol{\xi}_2/\boldsymbol{\xi}_0 \sim O(\epsilon^2)$. Subsequently, it is found that

$$\delta W = \delta W_0 + \delta W_2 + \delta W_4.$$

δW_0, δW_2 and δW_4 are minimised with respect to $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ as shown in the following.

Minimisation of δW_0

We initially consider the plasma region, i.e. $\delta W_\perp$ defined in Eq. (3.20). In both δW_0 and δW_2 we use (see exercise series)

$$\boldsymbol{\xi}_\perp = \xi_r \boldsymbol{\nabla} r + r\xi_{\perp\theta}\boldsymbol{\nabla}\theta + R\xi_{\perp\phi}\boldsymbol{\nabla}\phi$$

$$\xi^r = \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} r, \quad \xi_\perp^\theta = r\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\theta, \quad \xi_\perp^\phi = R\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\phi.$$

$$\boldsymbol{\delta B}_\perp = \delta B_r \boldsymbol{\nabla} r + r\delta B_{\perp\theta}\boldsymbol{\nabla}\theta + R\delta B_{\perp\phi}\boldsymbol{\nabla}\phi$$

$$\delta B^r = \boldsymbol{\delta B}_\perp \cdot \boldsymbol{\nabla} r, \quad \delta B_\perp^\theta = r\boldsymbol{\delta B}_\perp \cdot \boldsymbol{\nabla}\theta, \quad \delta B_\perp^\phi = R\boldsymbol{\delta B}_\perp \cdot \boldsymbol{\nabla}\phi.$$

We will later show (also exercises) that $\delta B_{\perp 0} \sim \epsilon B_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_{\perp 0}$. Since also $P \sim \epsilon^2 B^2$, only the second term in the convenient expression Eq. (3.20) for δW appears at leading order:

$$\delta W_0 = \frac{1}{2}\int d^3x B^2 \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp 0} + 2\boldsymbol{\xi}_{\perp 0}\cdot\boldsymbol{\kappa}\right|^2. \tag{3.27}$$

Since this term is positive definite the minimisation of Eq. (3.27) corresponds to $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp 0} + 2\boldsymbol{\xi}_{\perp 0}\cdot\boldsymbol{\kappa} = 0$, which gives $\delta W_0 = 0$. The leading order displacement does not allow poloidal coupling; $\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0(r)e^{i(n\phi - m\theta - \omega t)}$. With this and $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp 0} + 2\boldsymbol{\xi}_{\perp 0}\cdot\boldsymbol{\kappa} = 0$ one has (see exercises):

$$\xi_{\perp 0}^\theta = -\frac{i}{m}\frac{\partial}{\partial r}\left(r\xi_0^r\right). \tag{3.28}$$

We have seen that magnetic field compression energy vanishes at order zero. Since $\delta W_0 = 0$, the next order contribution δW_2 must be considered.

Minimisation of δW_2

All except the third term in Eq. (3.20) feature in δW_2. This term is neglected because the ordering of $\beta \sim O(\epsilon^2)$ yields that terms involving the equilibrium pressure first appear in δW_4. Referring to Eq. (3.20) it is now clear that

$$\delta W_2 = \frac{1}{2}\int d^3x \left(|\delta B_{\perp 0}|^2 + B^2 \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp 1} + 2\boldsymbol{\xi}_{\perp 1}\cdot\boldsymbol{\kappa}\right|^2 - J_{\parallel}(\boldsymbol{\xi}^*_{\perp 0}\times\boldsymbol{b})\cdot\boldsymbol{\delta B}_{\perp 0}\right). \qquad (3.29)$$

The second term is minimised to zero by a constraint on $\boldsymbol{\xi}_{\perp 1}$ such that,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp 1} + 2\boldsymbol{\xi}_{\perp 1}\cdot\boldsymbol{\kappa} = 0. \qquad (3.30)$$

Meaning that magnetic field compression vanishes also at second order. Hence $\boldsymbol{\xi}_{\perp 1}$ is eliminated from the problem at order δW_2, which is convenient, because $\boldsymbol{\xi}_{\perp 1}$ comprises poloidal harmonics in a torus, each harmonic satisfying (see exercises):

$$\frac{\partial}{\partial\theta}\xi_1^\theta + \frac{\partial}{\partial r}\left(r\xi_1^r\right) = 0.$$

It is now clear that δW (and associated growth rates) calculated to order ϵ^2 are the same in a torus and a cylinder (screw pinch).

For minimising δW_2 we require $\boldsymbol{\delta B}_{\perp 0}$ in terms of ξ_0^r. The perturbed field is defined in terms of the displacement via Eq. (3.3). As shown in the exercises:

$$\begin{aligned}\delta B_0^r &= \frac{imB_0}{R_0}\left(\frac{n}{m}-\frac{1}{q}\right)\xi_0^r \\ \delta B_0^\theta &= \frac{B_0}{R_0}\frac{\partial}{\partial r}\left[\left(\frac{n}{m}-\frac{1}{q}\right)r\xi_0^r\right].\end{aligned} \qquad (3.31)$$

which is sufficient to define $\boldsymbol{\delta B}_\perp$ and $|\delta B_\perp|$ to the necessary order in δW_2. In addition, from Ampère's law:

$$J_\parallel \approx J_\phi = \frac{1}{r}\frac{d}{dr}(rB_p) \approx \frac{B_0}{R_0}\frac{1}{r}\frac{d}{dr}\left(\frac{r^2}{q}\right). \qquad (3.32)$$

Minimisation of δW_2

Hence, using $\int d^3x \approx R_0 \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \int_0^a r\,dr$ we have simply,

$$\delta W_2 = \frac{2\pi^2 B_0^2}{R_0} \int_0^a dr \left\{ r \left[m^2 \left(\frac{n}{m} - \frac{1}{q} \right)^2 |\xi_0^r|^2 + \left(\frac{d}{dr} \left[\left(\frac{n}{m} - \frac{1}{q} \right) r |\xi_0^r| \right] \right)^2 \right] + \left(\frac{d}{dr} \left[\frac{r^2}{q} \right] \right) \left[\left(\frac{n}{m} - \frac{1}{q} \right) |\xi_0^r| \frac{d}{dr} (r|\xi_0^r|) + |\xi_0^r| \frac{d}{dr} \left[\left(\frac{n}{m} - \frac{1}{q} \right) r |\xi_0^r| \right] \right] \right\}.$$

Replacing ξ_0^r with $\xi_0^r / \exp(-im\theta + in\phi - i\omega t)$ so that we may drop the modulus in $|\xi_0^r|$, and after some integration by parts and cancellation, Eq. (3.29) becomes,

$$\delta W_2 = \frac{2\pi^2 B_0^2}{R_0} \left\{ \int_0^a dr\, r \left[\left(r \frac{d\xi_0^r}{dr} \right)^2 + \left(m^2 - 1 \right) (\xi_0^r)^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2 + a^2 (\xi_0^r(a))^2 \left[\frac{2}{q_a} \left(\frac{n}{m} - \frac{1}{q_a} \right) + \left(\frac{n}{m} - \frac{1}{q_a} \right)^2 \right] \right\} \tag{3.33}$$

where $q_a = q(a)$. Equation (3.33) shows that to second order in the inverse aspect ratio, internal modes ($\delta B^r(a) = 0$, $\xi^r(a) = 0$) are stable to ideal MHD since all the terms in the plasma region are positive or zero.

- Surface corrections (such that perturbed magnetic fields will extend into the vacuum) also appear at order ϵ^2, and these can be destabilising. This is the only source of destabilisation at order ϵ^2 in the ideal model.
- Resistive corrections also introduce possibility of destabilisation at order ϵ^2 (see later)
- At second order in δW, toroidal corrections determine the eigenfunction $\xi_{\perp 1}$, but the stability problem (also growth rate - see later) eliminates $\xi_{\perp 1}$ in such a way that its effect is irrelevant. Hence the stability of a cylinder and a torus is identical at order ϵ^2.
- Hence ideal external kinks and resistive tearing modes, which can be unstable at order ϵ^2, can often be legitimately treated under the cylindrical approximation (i.e. single harmonic, no toroidicity). Most NTM analytic modelling is based on cylindrical treatment.
- Ideal internal modes must calculate δW to order ϵ^4, where toroidal effects are crucial.

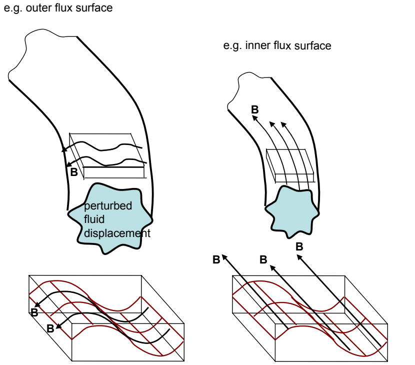
Field line bending stabilisation

Consider internal modes conforming to Eq. (3.33) for which we force $\delta B(a) = 0$. We will examine whether this is a valid boundary condition. First, the integrand of Eq. (3.33) is positive definite, meaning that internal modes cannot be unstable at this order in δW. However, we see that marginal stability occurs when $q(r) = m/n$ in all the plasma.

For negligible magnetic shear the stabilisation associated with the bending of field lines, in response to a plasma displacement, vanishes when the pitch of the unperturbed magnetic field lines is perpendicular to the wave vector (see figure), i.e. when

$$|\delta B_\perp| \propto k_\parallel B = 0 \approx \left(n - \frac{m}{q}\right) \frac{B_0}{R_0}$$

Recall the "frozen in theorem", which should help in understanding the figure. Bending field lines requires energy ($\int d^3x\, |\delta B_\perp|^2/2$), and this strongly damps instabilities. Instabilities occur when and where field line bending is not strong (where $q(r) \approx m/n$).



k is parallel to B, i.e. k_par is non-zero
Frozen in theorem means that field lines have to bend with fluid motion

k is perpendicular to B, i.e. k_par =0.
Field lines move, due to frozen In theorem, but they don't bend.

Internal modes

We minimise the energy of Eq. (3.33) by applying the Euler-Lagrange minimisation, as defined in Eq. (3.14), giving

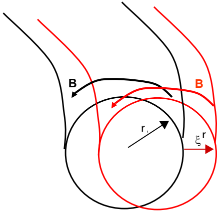
$$\frac{d}{dr}\left[\left(\frac{n}{m}-\frac{1}{q}\right)^2 r^3 \frac{d\xi_0^r}{dr}\right] = (m^2-1)\left(\frac{n}{m}-\frac{1}{q}\right)^2 r\xi_0^r. \quad (3.34)$$

We note that for $n = 0$ this turns out to be the same equation as Eq. (2.26) defining the equilibrium shaping terms S_m.

Equation (3.34) is singular for $q = m/n$. Near the magnetic axis q is approximately constant, so that the exact solution for small r is $\xi_0^r(r) \sim r^{-1\pm m}$, where the regular solution at $r = 0$ is clearly:

$$\xi_0^r \sim r^{m-1} \quad \text{and} \quad \xi_0^\theta \sim -(i/m)\xi_0^r.$$

For $m > 1$ the mode amplitude vanishes at the plasma centre. At the edge, under certain conditions, modes can be driven unstable by the magnetic field that extends into the vacuum region. These external kink modes will be considered later.



m=1 fluid displacement just shifts the field lines onto displaced torus. As seen, field lines don't bend in poloidal plane in response to m=1.

The $m = 1$ mode is a special case. Magnetic field line bending stabilisation (see figure for explanation) vanishes for $\xi_0^r \sim r^{m-1}$, since $(\xi_0^r)' = 0$ and $m^2 - 1 = 0$. This solution is valid near the magnetic axis. In order to satisfy the boundary condition $\xi_0^r(a) = 0$, the displacement can reduce δW_2 to zero by being constant in the core region where $q < 1$ (assuming $n = 1$), and zero in the region where $q > 1$. There is a narrow transition region, or layer region, which requires consideration of the effect of inertia (later). Since δW_2 vanishes, we need to go to δW_4. At this order, toroidal effects, including the Shafranov shift, are crucial. The effects of inertia, and toroidal effects on the $m = n = 1$ internal kink mode will be considered in detail later.

$m = 1$ internal kink mode

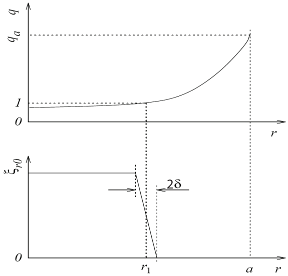
Both δW_0 and δW_2 have been minimised to zero. That is, the internal kink mode is marginally stable to order ϵ^2. The stability of the kink mode therefore depends on the fourth order correction, i.e. since both δW_0 and δW_2 have been minimised to zero one must consider fourth order terms. In calculating δW_4 the results for $\boldsymbol{\xi}_{\perp 0}$ and $\boldsymbol{\xi}_{\perp 1}$ given in the previous two sections must be used. The remaining component of $\boldsymbol{\xi}_{\perp 1}$ together with $\boldsymbol{\xi}_{\perp 2}$ must be identified.

A cylindrical calculation accurate to fourth order in the aspect ratio was first published by Rosenbluth (1973). However, for the all important $n = 1$ case, Bussac *et al* (1976) demonstrated that the fourth order cylindrical calculation is irrelevant. By including the toroidal effects of Shafranov shifted circular flux surfaces, Bussac showed that the fourth order solution has the form

$$\delta \hat{W}_4 = \left(1 - q_s^2\right) \delta \hat{W}_4^C + q_s^2 \delta \hat{W}_4^T \qquad (3.35)$$

$$\text{with } \delta W = \delta \hat{W} (2\pi^2 R_0 B_0^2 |\xi_0^r|^2 \epsilon^4)\Big|_{r_1 - \delta}$$

where $q_s = m/n$ ($= 1/n$ for $m = 1$), $\delta \hat{W}_4^C$ is the cylindrical potential energy [Rosenbluth] and $\delta \hat{W}_4^T$ is a toroidal contribution.



Notes

Toroidal and cylindrical δW

With $q_s = m^2/n^2$ and $m = 1$:

$$\delta\hat{W} = (1 - q_s^2)\delta\hat{W}_C + q_s^2\delta\hat{W}_T$$

with

$$\delta\hat{W}_C = -\frac{1}{\epsilon_s q_s^4}\int_0^{r_s}\frac{dr\, r^2}{r_s^3}\alpha$$

$$\delta\hat{W}_T = -\frac{1}{\epsilon_s^2 q_s^4}\left\{\left[\frac{3+c}{1-c}\right]\left[\int_0^{r_s}\frac{dr\, r^2}{r_s^3}\alpha\right]^2 - \frac{13}{4}\epsilon_s^2\int_0^{r_s}\frac{dr\, r^3}{r_s^4}q_s\left(\frac{1}{q} - \frac{1}{q_s}\right)\right\},$$

where,

$$\alpha = -\frac{2q_s^2 R_0}{B_0^2}\frac{dP}{dr}$$

so that

$$\beta_p = -2\frac{q(r_s)^2}{B_0^2\epsilon_s^2}\int_0^{r_s}\frac{dr\, r^2}{r_s^2}\frac{dP}{dr} = \frac{1}{\epsilon_s}\int_0^{r_s}\frac{dr\, r^2}{r_s^3}\alpha.$$

Also c has to do the upper sideband. Analytical solutions are possible. For example, for

$$q = q_s + \Delta[(r/r_s)^\nu - 1],$$

for $q_s = 1$ we obtain $c = -3 + 12\nu\Delta/(4-\nu)$. The well known result $\delta\hat{W}_4^T \approx 3(1-q_0)[(\beta_p^c)^2 - \beta_p^2]$ is obtained for $\nu = 2$ and $\Delta = 1 - q_0$.

$m = n = 1$ internal kink mode

EPFL

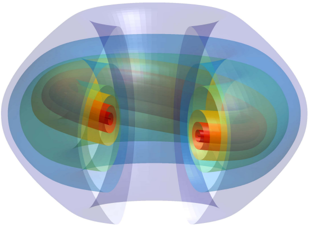
It is seen that the cylindrical term is entirely cancelled (for $n = 1$) by the inclusion of toroidal effects!! This was probably a massive surprise, and made the community realise that inclusion of full toroidal effects is not simply important, but crucial, even in the limit of infinite aspect ratio. The $m = n = 1$ mode generates a new tilted toroidal equilibrium. The field lines lie on new toroidal surfaces which are shifted relative to the initial equilibrium by ξ^r.

Such is its complexity, the calculation for δW_4 has only been performed analytically by a few researchers, and is beyond the scope of this course. Toroidal effects will be uncovered in more detail in this course by considering interchange and ballooning modes (these modes are easier to treat).

For a simple quadratic q profile:

$$\delta \hat{W}_4^T \approx 3(1 - q_0)\left((\beta_p^c)^2 - \beta_p^2\right), \qquad (3.36)$$

where $\beta_p = \beta_p(r_1)$, and typically $0.1 < \beta_p^c < 0.3$. So it is seen that the internal kink mode is an ideal mode, with instability governed by the q-profile, and pressure profile.



The $n = m = 1$ internal kink mode produces a rigid shifted torus, with the field line surfaces displaced from the original toroidal surfaces by amplitude ξ^r. Due to the radial dependence of ξ^r, the shift occurs only in the core region where $q < 1$.