# Plasma Instabilities

## Solutions for Exercises Series 7

*Fast ion effects on pressure driven long wavelength instabilities*

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1. It is useful to make an initial change of variables, writing as :

$$n_h = 2\pi \sum_\sigma \int_0^\infty dv_\parallel \int_0^\infty dX_\perp F_h$$

where $\sigma$ is the sign of $v_\parallel$ and

$$X_\perp = \frac{v_\perp^2}{2}.$$

The density is then,

$$n_h = 2\pi \sum_\sigma \int_0^\infty d\mathcal{E} \int_0^{1/B} d\lambda \, J \, F_h,$$

where we note the limits of integration follow from the discussion at the start of the question sheet, and the Jacobian of transformation is,

$$J = \left| \frac{\partial v_\parallel}{\partial \mathcal{E}} \frac{\partial X_\perp}{\partial \lambda} - \frac{\partial v_\parallel}{\partial \lambda} \frac{\partial X_\perp}{\partial \mathcal{E}} \right|.$$

Due to integration range in $\lambda$, we no longer need to concern ourselves with the sign of $v_\parallel$ in the following definitions:

$$v_\parallel = \left( \frac{2\mathcal{E}}{m_h} \right)^{1/2} \sqrt{1 - \lambda B}, \qquad X_\perp = \frac{\mathcal{E} \lambda B}{m_h}.$$

We obtain

$$\begin{aligned} J &= \left| \frac{\partial v_\parallel}{\partial \mathcal{E}} \frac{\partial X_\perp}{\partial \lambda} - \frac{\partial v_\parallel}{\partial \lambda} \frac{\partial X_\perp}{\partial \mathcal{E}} \right| \\ &= \left| \left( \left( \frac{2\mathcal{E}}{m_h} \right)^{1/2} (2\mathcal{E})^{-1} \sqrt{1 - \lambda B} \right) \left( \frac{\mathcal{E} B}{m_h} \right) - \left( \left( \frac{2\mathcal{E}}{m_h} \right)^{1/2} (-B/2) \frac{1}{\sqrt{1 - \lambda B}} \right) \left( \frac{\lambda B}{m_h} \right) \right| \\ &= \left| \frac{1}{2m_h} \left( \frac{2\mathcal{E}}{m_h} \right)^{1/2} \frac{B}{\sqrt{1 - \lambda B}} \left[ (1 - \lambda B) + B\lambda \right] \right| \\ &= \left| \frac{\mathcal{E} B}{m_h^2 v_\parallel} \right| \end{aligned}$$

Hence,

$$\int_0^\infty dv_\parallel \int_0^\infty dX_\perp = \frac{1}{m_h^2} \int_0^\infty d\mathcal{E} \, \mathcal{E} \int_0^{1/B} d\lambda \frac{B}{|v_\parallel|}$$

We thus obtain the required result,

$$n_h = \frac{2\pi}{m_h^2} \sum_\sigma \int_0^\infty d\mathcal{E} \, \mathcal{E} \int_0^{1/B} \frac{d\lambda \, B}{|v_\parallel(\mathcal{E}, \lambda)|} F_h(\mathcal{E}, \lambda, \sigma, r).$$

2. This question simply notes that the energy and pitch angle integrals de-nest when the distribution is isotropic. Using the definition of $v_\parallel$ and noting that $F_h$ is independent of $\lambda$ and $\sigma$ for an isotropic distribution function, one obtains:

$$n_h = \frac{2^{3/2}\pi}{m_h^{3/2}} \left( \int_0^\infty d\mathcal{E}\, \mathcal{E}^{1/2} F_h(\mathcal{E}, r) \right) \left( \int_0^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \right)$$

where we have replaced $\sum_\sigma$ with 2.

Also, for the trapped density we note the comment at the start of the question sheet, in particular that at a given position $(r, \Theta)$, with corresponding magnetic field $B(r, \Theta)$, trapped particles have pitch angles that fall in the range,

$$\frac{1}{B_{max}} \leq \lambda \leq \frac{1}{B(r, \Theta)}.$$

Hence for the calculation of the trapped density the lower limit of integration in $\lambda$ must be changed relative to the calculation for the full density (where the lower limit is $\lambda = 0$):

$$n_t = \frac{2^{3/2}\pi}{m_h^{3/2}} \left( \int_0^\infty d\mathcal{E}\, \mathcal{E}^{1/2} F_h(\mathcal{E}, r) \right) \left( \int_{1/B_{max}}^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \right)$$

Note that for a general distribution that is not isotropic, the trapped density is

$$n_t = \frac{2^{1/2}\pi}{m_h^{3/2}} \int_0^\infty d\mathcal{E}\, \mathcal{E}^{1/2} \int_{1/B_{max}}^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \sum_\sigma F_h(\mathcal{E}, \lambda, \sigma, r)$$

and the total density is,

$$n_h = \frac{2^{1/2}\pi}{m_h^{3/2}} \int_0^\infty d\mathcal{E}\, \mathcal{E}^{1/2} \int_0^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \sum_\sigma F_h(\mathcal{E}, \lambda, \sigma, r)$$

3. Since the integrals have been de-nested (due to isotropy) and since the distribution function is not inside the pitch angle integral, the energy integrals cancel in the definition of the trapped fraction:

$$f_t = \frac{n_t}{n_h} = \left( \int_{1/B_{max}}^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \right) \Big/ \left( \int_0^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \right).$$

The analytic result for the trapped fraction simply follows from the identity,

$$\int d\lambda\, (1-\lambda B)^{-1/2} = -2\sqrt{1-\lambda B}$$

and then a little algebra gives

$$f_t = \left(1 - \frac{B}{B_{max}}\right)^{1/2}.$$

The passing fraction can be obtained by integrating over passing pitch angle space

$$f_p = \frac{n_p}{n_h} = \left( \int_0^{1/B_{max}} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \right) \Big/ \left( \int_0^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}} \right).$$

or we may obtain the result from the trapped fraction, i.e. $f_p = 1 - f_t$.

4. From $B = B_0(1 - \epsilon\cos\Theta)$ we have that the maximum value of $B$ on a given flux surface $r$ is at $\Theta = \pi$. So, $B_{max}(r) = B_0(1+\epsilon)$. Substituting these identities into the trapped fraction, and keeping only the leading order term in $\epsilon$, we have

$$f_t = \epsilon^{1/2}(1 + \cos\Theta)^{1/2}.$$

Note that text books usually quote the trapped fraction at $\Theta = 0$, for which $f_t(\Theta = 0) = \sqrt{2\epsilon}$. The quantity $\sqrt{2\epsilon}$ is often presented as *the trapped fraction*, probably, because all trapped particles in a large aspect ratio tokamak pass though $\Theta = 0$, so the trapped fraction at $\Theta = 0$ is where there are the most trapped particles. Or it might be because some texts and lecture notes incorrectly assume that trapped particles, at any position in the plasma, have pitch angles that fall in the range $1/B_{max} < \lambda < 1/B_{min}$. The correct range at a given position $(r, \Theta)$ with corresponding field strength $B(r, \Theta)$ is $1/B_{max} < \lambda < 1/B(r, \Theta)$. A more appropriate trapped fraction which is independent of poloidal angle would be

$$\overline{f_t} \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\, f_t = \frac{2}{\pi}\sqrt{2\epsilon}$$

5. The trapped and passing fractions are plotted below for $\epsilon = 0.1$

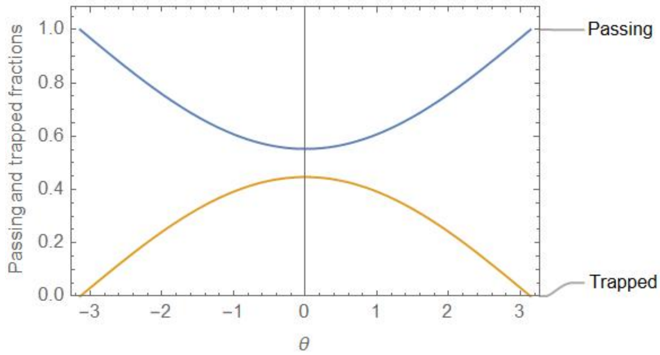


FIG. 1: Showing the passing and trapped fractions as a function of poloidal angle, assuming $\epsilon = 0.1$

At $\Theta = 0$ we have the largest trapped fraction and the smallest passing fraction. Particles that have turning points near $\Theta = 0$ are called deeply trapped particles. But we note that all trapped particles pass through $\Theta = 0$, even the ones that bounce near $\Theta = \pi$. The passing fraction $f_p = 1 - f_t$ is clearly least at $\Theta = 0$. There will be less and less trapped particles as we increase $\Theta$, some trapped particles bouncing before reaching the considered value of $\Theta$. The trapped fraction vanishes at $\Theta = \pi$. Such particles, if they were to exist, would spend an infinite time stuck at that point (analogous to a pendulum stuck vertically upwards). Particles with turning points near the limiting value $\Theta = \pi$ are called barely trapped particles, there are very few of these if the distribution is isotropic. Clearly at $\Theta = \pi$ the passing fraction $f_p = 1 - f_t$ is unity.

6. By making the requested transformation we easily obtain the result in the lectures:

$$\begin{aligned} F_h(\mathcal{E}, \mu, r) &= \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{m_h v_\parallel^2(B_{min})}{2T_\parallel(r)} - \frac{m_h v_\perp^2(B_{min})}{2T_\perp(r)}\right) \\ &= \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{\mathcal{E} - \mu B_{min}(r)}{T_\parallel(r)} - \frac{\mu B_{min}(r)}{T_\perp(r)}\right). \end{aligned}$$

7. Use the identities (from conservation of energy and magnetic moment) given in the question, which yield,

$$v_\perp^2(B_{min}) = \frac{B_{min}}{B} v_\perp^2$$

and

$$v_\parallel^2(B_{min}) = v_\parallel^2 + v_\perp^2(B_{min})\left(\frac{B}{B_{min}} - 1\right)$$

$$= v_\parallel^2 + v_\perp^2\left(1 - \frac{B_{min}}{B}\right).$$

The result is then easily obtained,

$$F_h = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{m_h v_\parallel^2}{2T_\parallel(r)} - \frac{m_h v_\perp^2}{2\hat{T}_\perp(r,\Theta)}\right),$$

$$\frac{1}{\hat{T}_\perp(r,\Theta)} = \frac{1}{T_\perp(r) B(r,\Theta)} \left[B_{min}(r) + \frac{T_\perp(r)}{T_\parallel(r)} \left(B(r,\Theta) - B_{min}(r)\right)\right].$$

8. Letting $v_\parallel = x$ and $v_\perp^2 = y$ the velocity integral is,

$$\int_{allV} d^3v = 2\pi \int_0^\infty dx \int_0^\infty dy$$

and the modified b-Maxwellian is

$$F_h = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{m_h x^2}{2T_\parallel(r)} - \frac{m_h y}{2\hat{T}_\perp(r,\Theta)}\right).$$

The integrals de-nest, so we have,

$$n_h = \frac{m_h^{3/2} n_c}{(2\pi)^{1/2} T_\perp T_\parallel^{1/2}} \left[\int_0^\infty dx \exp\left(-\frac{m_h x^2}{2T_\parallel}\right)\right] \left[\int_0^\infty dy \exp\left(-\frac{m_h y}{2\hat{T}_\perp}\right)\right].$$

Then use,

$$\int_0^\infty dz \exp\left(-C_1 z\right) = \frac{1}{C_1}, \quad \int_0^\infty dz \exp\left(-C_2 z^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{C_2}}$$

The required result:

$$n_h = n_c(r) \frac{\hat{T}_\perp}{T_\perp}.$$

is easily obtained on noting that

$$\frac{1}{C_1} = \frac{2\hat{T}_\perp}{m_h} \quad \frac{1}{2}\sqrt{\frac{\pi}{C_2}} = \sqrt{\frac{\pi T_\parallel}{2m_h}}.$$

9. First we write $1/\hat{T}_\perp$ in the form,

$$\frac{1}{\hat{T}_\perp} = \frac{1}{T_\perp}\left[1 + \left(\frac{B_{min}}{B} - 1\right)\left(1 - \frac{T_\perp}{T_\parallel}\right)\right].$$

We consider $B_{min}/B - 1 \sim \epsilon(\cos\Theta - 1)$ a small parameter, while $T_\perp/T_\parallel - 1$ may not be small in general. We thus have approximately,

$$\frac{\hat{T}_\perp}{T_\perp} \approx 1 + \left(\frac{B_{min}}{B} - 1\right)\left(\frac{T_\perp}{T_\parallel} - 1\right)$$

and then using leading order $B = B_0(1 - \epsilon\cos\Theta)$ and $B_{min} = B_0(1-\epsilon)$, and thus $B_{min}/B - 1 \sim \epsilon(\cos\Theta - 1)$ we obtain,

$$\frac{\hat{T}_\perp}{T_\perp} \approx \left[1 + \left(\frac{T_\perp}{T_\parallel} - 1\right)\epsilon(\cos\Theta - 1)\right].$$

Hence we easily obtain,

$$n_h(r,\Theta) = n_c(r)\frac{\hat{T}_\perp}{T_\perp} = n_c(r)\left[1 + \left(\frac{T_\perp}{T_\parallel} - 1\right)\epsilon\left(\cos\Theta - 1\right)\right].$$

The question asks us to show how anisotropy affects the density on the high field side and low field side. This is easily done by noting that $T_\perp/T_\parallel - 1 > 0$ for perpendicular anisotropy, for which the density is higher on the LFS (small $\Theta$ for which $\cos\Theta - 1 \approx 0$) than the HFS (for which $\cos\Theta - 1 \approx -2$) due to the enhanced trapped fraction. The opposite argument follows for parallel anisotropy.

10. For the isotropic case the density is independent of $\Theta$. Hence the density weighted average curvature is zero, at least for the lowest order radial curvature defined in the question. We can in fact analytically calculate the lowest order density weighted average curvature:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta \left(-\frac{1}{R_0}\cos\Theta\right) n_c(r)\left[1+\left(\frac{T_\perp}{T_\parallel}-1\right)\epsilon\,(\cos\Theta-1)\right] \\
=&\frac{n_c(r)}{R_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\,\epsilon\left(1-\frac{T_\perp}{T_\parallel}\right)(\cos\Theta-1)\cos\Theta \\
=&\frac{n_c(r)}{R_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\,\epsilon\left(1-\frac{T_\perp}{T_\parallel}\right)\cos^2\Theta \\
=&\frac{n_c(r)}{2R_0}\,\epsilon\left(1-\frac{T_\perp}{T_\parallel}\right).
\end{aligned}$$

Hence we see that the density weighted average curvature is zero for the isotropic case, it is positive for the parallel anisotropic case $T_\parallel > T_\perp$, and negative for the perpendicular anisotropic case $T_\parallel < T_\perp$. Concerning stabilisation or destabilisation for each case, read the comments at the end of the question (and look in the lecture notes).

11. This question is similar to the last question, but this time we evaluate,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\,\boldsymbol{\kappa}_0\cdot\boldsymbol{\nabla} r\, n_h(pass) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\left(-\frac{1}{R_0}\cos\Theta\right) n_h(pass).$$

with

$$n_h(pass) = n_c(r)\left[1-\epsilon^{1/2}(1+\cos\Theta)^{1/2}\right].$$

Hence we evaluate,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\left(-\frac{1}{R_0}\cos\Theta\right) n_c(r)\left[1-\epsilon^{1/2}(1+\cos\Theta)^{1/2}\right] \\
=&\frac{\epsilon^{1/2} n_c(r)}{R_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\,(1+\cos\Theta)^{1/2}\cos\Theta \\
=&\frac{n_c(r)}{R_0}\frac{2}{3\pi}(2\epsilon)^{1/2}
\end{aligned}$$

Taking the trapped fraction as $f_t = \epsilon^{1/2}(1+\cos\Theta)^{1/2}$ we note that the average trapped fraction is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\,\epsilon^{1/2}(1+\cos\Theta)^{1/2} = \frac{2}{\pi}(2\epsilon)^{1/2}.$$

Hence we find that the density weighted average curvature is:

$$\frac{n_c(r)}{R_0}\frac{\overline{f_t}}{3}$$

as requested in the question. As shown in the lecture notes, it is possible to obtain the passing density averaged curvature for the bi-Maxwellian. For example, the trapped and passing fractions can be calculated exactly for anisotropic cases, and expanded for small $\epsilon$.