Plasma Instabilities

Exercises Series 7

Fast ion effects on pressure driven long wavelength instabilities

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Passing and trapped fractions for isotropic distributions

Consider the constants of single particle motion in the absence of an electric field and collisions: $\mathcal{E} = m_h v^2/2 = m_h v_\parallel^2/2 + m_h v_\perp^2/2$, and pitch angle $\mu = m_h v_\perp/(2B)$, $\lambda = \mu/\mathcal{E}$ and $\sigma = \text{sign}(v_\parallel)$. The radius r is also considered a constant of motion (this is true in the thin banana limit). We may define a pitch angle $\lambda = \mu/\mathcal{E}$ which is also clearly a constant of motion. We may now write $v_\parallel$ in terms of these constants of motion and the varying quantity $B(\Theta)$ (assuming a tokamak so no ϕ dependendence):

$$v_\parallel(\Theta) = \pm \left(\frac{2\mathcal{E}}{m_h}\right)^{1/2} \sqrt{1 - \lambda B(\Theta)},$$

the plus and minus being important for identifying co and counter passing ions, or for defining the inner and outer legs of the trapped banana orbit. Passing particles are defined as those for which $1 - \lambda B$ (and hence $v_\parallel$) does not change sign as the particle explores all the values of B on a flux surface r (so over varying Θ in a tokamak along the whole length along the field line). Hence a particle is passing if it has a pitch angle in the range:

$$0 \leq \lambda < \frac{1}{B_{max}}.$$

Here $B_{max}(r) = B(r, \Theta = \pi)$ in a large aspect ratio tokamak. Trapped particles in contrast have pitch angles for which $1 - \lambda B$ does change sign as the particle undertakes its dominant motion along the field lines. Hence a particle will be trapped if it has a pitch angle in the range,

$$\frac{1}{B_{max}} \leq \lambda \leq \frac{1}{B_{min}}.$$

Here $B_{min}(r) = B(r, \Theta = 0)$ in a large aspect ratio tokamak. Now a point that is often a bit confused. It should be noted that $1/B_{min}$ is the largest value that λ can take, but it only applies to a deeply trapped particle that cannot escape from $B = B_{min}$, i.e. the outboard midplane in a large aspect ratio tokamak. But, if we choose to look at another position (r, Θ) with associated magnetic field $B = B(r, \Theta)$, the pitch angles for trapped particles at that location will fall in the range

$$\frac{1}{B_{max}} \leq \lambda \leq \frac{1}{B(r, \Theta)}.$$

This will be important for calculating the trapped fraction. Finally, after all of the above, the full range of λ is:

$$0 \leq \lambda \leq \frac{1}{B(r, \Theta)}.$$

1. Isotropic equilibrium distributions depend only on energy $\mathcal{E}$ and radial position r (in the limits of very small banana width). When dealing with isotropic distributions it is not convenient to use the velocity space integral (for the density) in the form:

$$n_h = \int_{allV} d^3v\, F_h = 2\pi \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp\, v_\perp F_h$$

With definitions $\mathcal{E} = m_h v_\parallel^2/2 + m_h v_\perp^2/2$, $\mu = m_h v_\perp/(2B)$, $\lambda = \mu/\mathcal{E}$ and $\sigma = \text{sign}(v_\parallel)$, first for a general anisotropic distribution show that,

$$n_h = \frac{2\pi}{m_h^2} \sum_\sigma \int_0^\infty d\mathcal{E}\, \mathcal{E} \int_0^{1/B} \frac{d\lambda\, B}{|v_\parallel(\mathcal{E}, \lambda)|} F_h(\mathcal{E}, \lambda, \sigma, r),$$

with

$$|v_{\parallel}(\mathcal{E},\lambda)| = \left(\frac{2\mathcal{E}}{m_h}\right)^{1/2} \sqrt{|1-\lambda B|},$$

Be careful with the upper limit of λ, consider the discussion at the start of this exercise.

2. For an isotropic distribution, show that

$$n_h = \frac{2^{3/2}\pi}{m_h^{3/2}} \left(\int_0^{\infty} d\mathcal{E}\, \mathcal{E}^{1/2} F_h(\mathcal{E},r)\right) \left(\int_0^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}}\right)$$

and taking into account the notes at the start of this question sheet show that the density of trapped particles is:

$$n_t = \frac{2^{3/2}\pi}{m_h^{3/2}} \left(\int_0^{\infty} d\mathcal{E}\, \mathcal{E}^{1/2} F_h(\mathcal{E},r)\right) \left(\int_{1/B_{max}}^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}}\right)$$

3. Show that the fraction of trapped particles f_t is independent of the details of the isotropic distribution, in particular that

$$f_t = \frac{n_t}{n_h} = \left(\int_{1/B_{max}}^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}}\right) \Big/ \left(\int_0^{1/B} \frac{d\lambda\, B}{\sqrt{1-\lambda B}}\right).$$

Perform the integration to show that,

$$f_t = \left(1 - \frac{B}{B_{max}}\right)^{1/2}$$

and the passing fraction f_p is (hint note that $f_p = 1 - f_t$)

$$f_p = 1 - \left(1 - \frac{B}{B_{max}}\right)^{1/2}$$

(Note that these results are exact providing the distribution function is isotropic, and the radial excursion (banana width) is not too large. It doesn't require axisymmetry. So it would be valid for example for the trapped fraction of alpha particles in a stellarator (caution - in practice fast particle distributions are rarely isotropic in a stellarator because trapped energetic ions are usually badly confined unfortunately - improving their confinement by stellarator optimisation is a major area of research).)

4. Apply the lowest order approximation for the magnetic field $B = B_0(1 - \epsilon\cos\Theta)$. What is $B_{max}(r)$ on a given flux surface? Show that to leading order in ϵ,

$$f_t = \epsilon^{1/2}(1+\cos\Theta)^{1/2}.$$

5. Sketch the passing and trapped fraction as a function of Θ. Where are each of the fractions largest and smallest, and why? What are the limiting values on a flux surface for each fraction?

Anisotropic Distributions

6. A valid distribution function for the fast ions should depend only on the constants of motion. Adapt a standard bi-Maxwellian

$$\frac{m_h^{3/2} n_h(r)}{(2\pi)^{3/2} T_{\perp}(r) T_{\parallel}(r)^{1/2}} \exp\left(-\frac{m_h v_{\parallel}^2}{2T_{\parallel}(r)} - \frac{m_h v_{\perp}^2}{2T_{\perp}(r)}\right)$$

by replacing $v_{\parallel}(\mathcal{E},\lambda,r,\Theta) \rightarrow v_{\parallel}(\mathcal{E},\lambda,r,\Theta_{min})$, and $v_{\perp}(\mathcal{E},\lambda,r,\Theta) \rightarrow v_{\perp}(\mathcal{E},\lambda,r,\Theta_{min})$ where $B_{min}(r) = B(r,\Theta_{min})$. Also replace $n_h(r) \rightarrow n_c(r)$, since the true density moment of the distribution function F_h has to be evaluated, in particular it isn't a flux function if $T_{\parallel} \neq T_{\perp}$.

7. Write the expression for the distribution obtained in the last question in terms of $v_\parallel(\mathcal{E},\lambda,r,\Theta)$, $v_\perp(\mathcal{E},\lambda,r,\Theta)$, $B(r,\Theta)$ and $B_{min}(r)$. First, use the conservation of $\mathcal{E}$ and μ (or λ) to obtain the identities:

$$\frac{v_\perp^2(B)}{B} = \frac{v_\perp^2(B_{min})}{B_{min}}$$

and

$$v_\parallel^2(B) = v_\parallel^2(B_{min}) + v_\perp^2(B_{min}) - \frac{v_\perp^2(B_{min})}{B_{min}}B.$$

Hence show that

$$F_h = \frac{m_h^{3/2} n_c(r)}{(2\pi)^{3/2} T_\perp(r) T_\parallel(r)^{1/2}} \exp\left(-\frac{m_h v_\parallel^2}{2T_\parallel(r)} - \frac{m_h v_\perp^2}{2\hat{T}_\perp(r,\Theta)}\right),$$

$$\frac{1}{\hat{T}_\perp(r,\Theta)} = \frac{1}{T_\perp(r)B(r,\Theta)}\left[B_{min}(r) + \frac{T_\perp(r)}{T_\parallel(r)}\left(B(r,\Theta) - B_{min}(r)\right)\right].$$

8. As shown in the lecture notes, calculating the moments of the modified bi-Maxwellian can be conveniently undertaken with:

$$\int_{allV} d^3v = 2\pi \int_0^\infty dv_\parallel \int_0^\infty dv_\perp^2.$$

Show that the density for the modified bi-Maxwellian is

$$n_h = n_c(r)\frac{\hat{T}_\perp}{T_\perp}.$$

Comment on whether the density depends on the poloidal angle when $T_\parallel = T_\perp$ and when $T_\parallel \neq T_\perp$. Explain these observations physically, based on your answers to questions 4 and 5.

9. Using the lowest order expression for the magnetic field $B = B_0(1 - \epsilon\cos\Theta)$ show that

$$n_h(r,\Theta) = n_c(r)\left\{1 + \left(\frac{T_\perp}{T_\parallel} - 1\right)\epsilon\left(\cos\Theta - 1\right)\right\}.$$

For $T_\parallel < T_\perp$ show that the density is larger on LFS (low field side, small Θ) than the HFS. Try to understand this intuitively by the fact that we have an enhanced fraction of trapped particles when the perpendicular temperature is larger than the parallel temperature. Similarly, For $T_\parallel > T_\perp$ show that the density is larger on HFS (high field side, $\Theta = \pi$) than the LFS. Try to understand this intuitively by the fact that we have a smaller fraction of trapped particles when the perpendicular temperature is smaller than the parallel temperature.

The weighted radial curvature

10. Consider the lowest order radial curvature,

$$\boldsymbol{\kappa}_0 \cdot \boldsymbol{\nabla} r = -\frac{1}{R_0}\cos\Theta$$

Now consider the average curvature weighted with the density:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\, n_h \boldsymbol{\kappa}_0 \cdot \boldsymbol{\nabla} r.$$

Show that it is zero for an isotropic distribution (so the next order in the curvature is needed, and that is indeed what is calculated in the core ideal MHD calculation - the Mercier terms), and show that it is negative for a distribution with $T_\parallel < T_\perp$, and positive for $T_\parallel > T_\perp$.

As shown in the notes having negative weighted average curvature is destabilising for core-peaked profiles, and this can be understood because the weight is largest in the region of bad curvature (LFS). Having positive weighted average curvature is stabilising for core-peaked profiles, and this can be understood because the weight is largest in the region of good curvature (HFS). These statements must be reversed if the density (or pressure) gradient is positive because the region of good curvature is in fact on the LFS for that particular population.

11. Consider now the lowest order passing density for an isotropic distribution

$$n_h(pass) = n_c(r)\left[1 - \epsilon^{1/2}(1+\cos\Theta)^{1/2}\right].$$

Now consider the average curvature weighted with the passing density:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\, n_h(pass)\boldsymbol{\kappa}_0\cdot\boldsymbol{\nabla} r.$$

Show that it is

$$\frac{n_c(r)}{R_0}\frac{\overline{f_t}}{3}$$

and that the average trapped fraction $\overline{f_t}$ is

$$\overline{f}_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\Theta\, f_t = \frac{2}{\pi}\sqrt{2\epsilon}$$

where the trapped fraction is approximately $f_t = \epsilon^{1/2}(1+\cos\Theta)^{1/2}$.

As shown in the lecture notes, for weakly collisional populations, it is in fact the passing particle weighted curvature that determines stability. That it is positive means that fast particles are stabilising for core peaked profiles. This can be understood because trapped particles, which exist in the region of poor curvature, don't contribute to the stability problem. Only passing particles contribute, and they spend more time in the region of good curvature. As seen, the passing particle density is larger on the HFS than on the low field side. The calculations can be repeated for an anisotropic distribution function, as has been undertaken in the lecture notes.