Plasma Instabilities

Exercises Series 6

Ballooning and interchange modes

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Obtaining ballooning equation in analytic equilibrium coordinates

1. The weird curvature is defined in the lecture notes as:

$$\kappa_w = \left(\frac{1}{B^2}\right)\left[\frac{\partial}{\partial\psi} + \left\{\frac{(\boldsymbol{\nabla}\beta)^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega - (\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\omega)}{B^2}\right\}\frac{\partial}{\partial\omega}\right]\left(\frac{B^2}{2}+P\right).$$

   Obtain the term in the {} brackets. Use from the start the simplification that one can neglect non-orthogonal contributions that arise from $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega$ when using the ordering,

$$r\Delta'' \sim \alpha \sim 1, \quad \Delta' \sim \epsilon, \quad s \sim 1$$

   where $\beta = \phi - q(\psi)\theta$, and θ is the straight field line angle. By using the result obtained in exercise series 2, $\theta = \omega - (\epsilon + \Delta')\sin\omega$, find that,

$$(\boldsymbol{\nabla}\beta)^2\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega - (\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\psi)(\boldsymbol{\nabla}\beta\cdot\boldsymbol{\nabla}\omega) = \frac{q^2(\psi')}{r^3}\left[(\epsilon + r\Delta'')\sin\omega - s\omega + O(\epsilon, \epsilon\cos\omega)\right]$$

   Note that the meaning of the $O()$ symbol in the last expression above is that it neglects all ϵ corrections except $\epsilon\sin\omega$. These are kept because they will be multiplied by $\partial B^2/\partial\omega \approx 2B_0^2\epsilon\sin\omega$ in κ_w, knowing that $\epsilon\sin^2\omega^2$ will provide a contribution for interchange modes. Convince yourself that non-orthogonal contributions from $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega$ can be neglected in the ballooning equation to relevant order.

2. To complete the calculation for κ_w use the total field (poloidal + toroidal - see lecture notes and exercise for week 2):

$$B^2 = \left(\frac{R_0}{R}\right)^2\left(1 + 2F_2 + \frac{\epsilon^2}{q^2}(1 + 2\Delta'\cos\omega)\right)$$

   with

$$\frac{dF_2}{dr} = -\frac{1}{B_0^2}\frac{dP}{dr} - \frac{r}{R_0^2q^2}(2-s).$$

   Hence obtain:

$$\kappa_w = -\frac{1}{\psi' R_0}\left[\cos\omega - \epsilon\left\{1 - \frac{1}{q^2}\right\} + \sin\omega(s\omega - r\Delta''\sin\omega) + O(\epsilon\sin\omega, \epsilon\cos\omega)\right].$$

3. Show next that (again dropping non-orthogonality)

$$(\boldsymbol{\nabla}\beta)^2 = \frac{q^2}{r^2}\left[1 + (s\omega - r\Delta''\sin\omega)^2 + O(\epsilon\sin\omega, \epsilon\cos\omega)\right]$$

   and together with κ_w from the last question, and using also (week 2) $\mathcal{J}_{r,\omega} = rR_0[1 - (\epsilon - \Delta')\cos\omega]$, show that the ballooning equation

$$\frac{1}{\mathcal{J}_{\psi,\omega}}\frac{\partial}{\partial\omega}\left[\left(\frac{\boldsymbol{\nabla}\beta}{B}\right)^2\frac{1}{\mathcal{J}_{\psi,\omega}}\frac{\partial}{\partial\omega}X\right] + 2\kappa_w\frac{dP}{d\psi}X = 0$$

can be reduced to

$$\frac{\partial}{\partial\omega}\left[\left\{1+(s\omega-\alpha\sin\omega)^2+O(\epsilon\sin\omega,\epsilon\cos\omega)\right\}\frac{\partial}{\partial\omega}X\right]+\alpha\left[\cos\omega-\epsilon\left\{1-\frac{1}{q^2}\right\}+\sin\omega(s\omega-\alpha\sin\omega)+O(\epsilon\sin\omega,\epsilon\cos\omega)\right]X=0 \tag{1}$$

in the limit $r\Delta'' \to \alpha$ (true to relevant order in ϵ if $\alpha \sim 1$)

4. Write down the lowest order (in ϵ) ballooning equation, assuming that $s \sim 1$ and $\alpha \sim 1$. Your result is the one used for obtaining the famous $s - \alpha$ stability diagram reproduced in the lecture slides.

Questions on obtaining interchange marginal stability

5. It is possible to write the infinite n ballooning equation for a general axisymmetric plasma in the form

$$\frac{d}{d\omega}\left[f\frac{dX}{d\omega}\right]+gX=0$$

with

$$f=a+b\omega+c\omega^2 \quad \text{and} \quad g=d+e\omega,$$

where a, b, c, d and e are 2π are periodic functions of ω.

For the analytically reduced ballooning equation (1) identify a, b, c, d and e.

6. To solve the interchange problem we look for the secular dependence of X in ω. We look for a solution of the form,

$$X=\omega^p\left[X_0(\omega)+\frac{X_1(\omega)}{\omega}+\frac{X_2(\omega)}{\omega^2}+O\left(\frac{X_3}{\omega^3}\right)\right] \tag{2}$$

where again, X_0, X_1 and X_2 are 2π periodic (no secular dependence in ω). The objective is to identify p but to do so it will be necessary to identify X_0, X_1 and X_2 (in fact X_2 can be eliminated in favour of p).

At this point do not adopt your expressions for a, b, c, d and e (keep them general, but remember that they are periodic in 2π). Solve the problem by substituting (2) into (1), then from sequential coefficients of the resulting polynomial find that,

$$X_0 = C, \quad \text{with C a constant}$$

$$\frac{dX_1}{d\omega}=-\left(p+\frac{\hat{e}}{c}\right)+\frac{p+\langle\hat{e}/c\rangle}{c\,\langle1/c\rangle}$$

$$\langle X_2\rangle=0 \quad \Longrightarrow \quad 0=(p+1)\left\{-\left(p\,\langle c\rangle+\langle\hat{e}\rangle\right)+\frac{p+\langle\hat{e}/c\rangle}{\langle1/c\rangle}\right\}+p(p+1)\,\langle c\rangle+\langle eX_1\rangle+\langle d\rangle\,,$$

where we have set $C = 1$. These results are obtained by performing integration with respect to ω, in particular, we have

$$\hat{e}=\int_0^\omega d\omega\, e, \quad \text{and} \quad \langle x\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\omega\, x.$$

7. Use the second of the above three equations to obtain $\langle\hat{e}X_1'\rangle$, where $' = d/d\omega$. Then use that $\langle(\hat{e}X_1)'\rangle = 0$, which follows because both $\hat{e}$ and X_1 are each periodic. Noting that $\langle(\hat{e}X_1)'\rangle = \langle eX_1\rangle + \langle\hat{e}X_1'\rangle = 0$, and using the just obtained result for $\langle\hat{e}X_1'\rangle$ and the third of the three equations above for $\langle eX_1\rangle$, show that

$$p=-\frac{1}{2}\pm\sqrt{\frac{1}{4}-D_M}, \quad \text{with, } D_M=\left\langle\frac{\hat{e}}{c}\right\rangle-\left\langle\frac{\hat{e}}{c}\right\rangle^2+\left\langle\frac{1}{c}\right\rangle\left(\left\langle\frac{\hat{e}\hat{e}}{c}\right\rangle+\langle d\rangle-\langle\hat{e}\rangle\right)$$

8. Show that for $D_M < 1/4$ the energy

$$\int_{-\infty}^{\infty} d\omega \mathcal{J}_\omega \left(-2\kappa_w \frac{dP}{d\psi} |X|^2 \right)$$

associated with one of the solutions is infinite. This unphysical result means that an interchange instability doesn't exist (equilibrium is stable to interchange modes).

9. Show that for $D_M \geq 1/4$ that the oscillatory eigenfunction has the form:

$$X = X_0 = A\omega^{-1/2} \exp\left[i|D_M - 1/4| \ln \omega\right] + B\omega^{-1/2} \exp\left[-i|D_M - 1/4| \ln \omega\right].$$

For the marginal case $D_M = 1/4$ show that in order for the energy to be finite we must have that $A = -B$. What element of physics should be added so that growth rates can be calculated, and so that physical eigenfunctions can be calculated?

10. For the analytic ballooning equation of Eq. (1) use your earlier identified parameters a, b, c, d, e, and obtain $\hat{e}$ and ultimately D_M.

Noting that c is a constant with respect to ω we may write the instability condition as:

$$D_M c > \frac{1}{4} c.$$

Explain what the $(1/4)c$ represents and what $D_M c$ represents. In your answer mention cylindrical curvature, toroidal curvature, and average curvature. What is the interchange instability condition in a cylinder? And in a reverse field pinch? Finally, what happens in an advanced tokamak scenario, operating with $q_{min} > 1$ (the magnetic shear reverses near the axis) in the region of weak shear if impurity transport causes α to be negative?

11. Show using the results of the previous questions that the same solution for X and the same marginal stability threshold is obtained from the *interchange equation* :

$$\frac{\partial}{\partial \omega} \left[\omega^2 \frac{\partial}{\partial \omega} X \right] + D_M X = 0.$$

Comment on which terms in the ballooning equation of Eq. (1) are redundant for interchange modes. Can interchange modes be described by the reduced ballooning equation you wrote down under question 4, i.e. the one used for making the ballooning diagram in the notes? Was it sufficient to use the reduced ballooning equation under question 4 in order to make the ballooning diagram for equilibria with $q \gg 1$ and α positive?

12. Why are ballooning modes, which conform with Eq. (1), more unstable than interchange modes? Are there regions where we expect tokamaks to be certainly stable to interchange modes, but possibly unstable to ballooning? Under what conditions can the tokamak be stable ballooning modes? As part of the discussion describe the reason for the second region of stability.