Plasma Instabilities

Solutions for Exercises Series 5

Linear and non-linear Tearing Modes

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1. On the separatrix we have $\Psi = r_s \hat{B}_1^r(r_s,t)/m$, so that,

$$x^2 = \frac{2\hat{B}_1^r(r_s,t)}{msB_0^\theta}\left[1-\cos(m\chi)\right].$$

Solutions to x are thus $x_+(\chi)$ and $x_-(\chi)$:

$$x_{+,-}(\chi) = \pm\sqrt{\frac{2\hat{B}_1^r(t)}{msB_0^\theta}|1-\cos(m\chi)|}$$

The maximum width occurs for $m\chi=\pi$ (see figure in the slides for $m=1$) for which $|1-\cos(m\chi)|=2$. The full width is therefore the difference between the two solutions at $m\chi=\pi$ multiplied by r_s (recall x is a normalised variable):

$$w = r_s\left[x_+(\chi=\pi/m) - x_-(\chi=\pi/m)\right] = 2r_s x_+(\chi=\pi/m) = 2r_s\sqrt{\frac{2\hat{B}_1^r(t)}{msB_0^\theta}2} = 4r_s\left(\frac{\hat{B}_1^r(t)}{msB_0^\theta}\right)^{1/2}.$$

2. Starting with

$$\int_{r_s-w/2}^{r_s+w/2} dr\frac{\partial\Psi_1}{\partial t} = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}$$

we use the constant-psi approximation by taking Ψ_1 outside the integral on the LHS. This means we set $\Psi(r,t)=\Psi(r_s,t)$ so that $\partial\Psi_1/\partial t = d\Psi_1(r_s,t)/dt$:

$$\int_{r_s-w/2}^{r_s+w/2} dr\frac{d\Psi_1(r_s,t)}{dt} = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}$$

so that

$$\frac{d\Psi_1(r_s,t)}{dt}w = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}$$

We now use the result of the previous question, in particular $\Psi_1(r_s,t) = Cw(t)^2$ where C is a constant. Hence,

$$\frac{d\Psi_1(r_s,t)}{dt}w = Cw\frac{dw(t)^2}{dt} = 2Cw^2\frac{dw(t)}{dt} = 2\Psi_1(r_s,t)\frac{dw(t)}{dt}.$$

So that

$$2\Psi_1(r_s,t)\frac{dw(t)}{dt} = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}.$$

We then easily obtain,

$$\frac{dw(t)}{dt} = \frac{\eta(r_s)}{2}\frac{1}{\Psi_1(r_s)}\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}$$

Noting time dependence on the RHS has been normalised out. Using constant-psi approximation for $1/\Psi_1(r_s)$ on the RHS of the above, we obtain the demanded result:

$$\frac{dw}{dt} = \frac{\eta(r_s)}{2}\Delta'(w) \quad \text{with} \quad \Delta'(w) = \frac{1}{\Psi_1}\frac{d\Psi_1}{dr}\bigg|_{r_s-w/2}^{r_s+w/2}.$$

Note that we could equally define,

$$\Delta'(w) = \frac{1}{\Psi_1(r_s)}\frac{d\Psi_1}{dr}\bigg|_{r_s-w/2}^{r_s+w/2}.$$

3. Follow the guidance in the question by allowing radial dependence only in Ψ_1''. Also, let $j_{BS} + j_{cd} = j_{non}$:

$$\frac{d\Psi_1(r_s)}{dt} = \eta(r_s)\left[\frac{\partial^2\Psi_1}{\partial r^2} + j_{non}(r_s)\right]$$

We then integrate as directed

$$\int_{r_s-w/2}^{r_s+w/2} dr\frac{\partial\Psi_1}{\partial t} = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2} + j_{non}(r_s)\int_{r_s-w/2}^{r_s+w/2} dr,$$

so that

$$\frac{d\Psi_1(r_s,t)}{dt}w = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2} + \eta(r_s)j_{non}(r_s)w$$

We now follow the approach in the last question, i.e. use $\Psi_1(r_s,t) = Cw(t)^2$ where C is a constant. Using the last question as a guide we use

$$\frac{d\Psi_1(r_s,t)}{dt}w = 2\Psi_1(r_s,t)\frac{dw(t)}{dt}.$$

This yields,

$$2\Psi_1(r_s,t)\frac{dw(t)}{dt} = \eta(r_s)\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2} + \eta(r_s)j_{non}(r_s)w$$

or

$$\begin{aligned}\frac{dw(t)}{dt} &= \frac{\eta(r_s)}{2}\frac{1}{\Psi_1(r_s)}\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2} + \frac{\eta(r_s)}{2}j_{non}(r_s)\frac{w}{\Psi_1(r_s)} \\ &= \frac{\eta(r_s)}{2}\frac{1}{\Psi_1(r_s)}\frac{\partial\Psi_1}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2} + \frac{\eta(r_s)}{2}j_{non}(r_s)\frac{1}{Cw}\end{aligned}$$

Clearly, we require a value for C. We know this from the first question in this series. From

$$w = 4r_s\left(\frac{\hat{B}_1^r(t)}{msB_0^\theta}\right)^{1/2}$$

and $\Psi_1(r_s) = r_sB_1(r_s)/m$ we obtain,

$$\frac{1}{C} = \frac{16r}{sB_0^\theta}\bigg|_{r_s}.$$

We therefore find that,

$$\frac{dw(t)}{dt} = \frac{\eta(r_s)}{2}\left[\Delta'(w) + \Delta'_{non}(w)\right] \quad \text{with} \quad \Delta'_{non}(w) = \left.\frac{j_{non}}{w}\frac{16r}{sB_0^\theta}\right|_{r_s}.$$

Clearly, $1/w$ diverges for $w \to 0$, so the literature replaces

$$\frac{1}{w} \to \frac{w}{w_c^2 + w^2}$$

with w_c a constant. Note that the non-linear calculation cannot recover the linear results because inertia was neglected, so the limit $w \to 0$ is ad-hoc in the non-linear treatment that gave the $1/w$ dependency. Of course, there should be no divergence, as the linear regime will not be singular in w. So that fact justifies the replacement. Of course, w_c should be much smaller than the saturated solution for w. Physically, w_c represents the lowest physical island width over which the pressure is effectively flattened. Pressure gradients can be established across small islands. Transport calculations are undertaken to establish w_c for a given plasma equilibrium.

It is perhaps worth mentioning that co-cd or counter-cd (current drive that is or co or counter to the Ohmic current) will produce respectively destabilising (increased w) or stabilising (reduced w) effects. Bootstrap current is also destabilising if the pressure gradient is negative (usually the case). Current drive from ECH is therefore used in experiments to control NTMs. Note that ICCD can also be used, but ECH and ECCD tend to be much more localised, potentially providing strong localised current drive. But this requires accurate tracking of the local of the mode in real time. ITER plans to dedicate some of its EC power for this purpose. Note that auxiliary heating (as opposed to current drive) can also control NTMs to some extent by locally adjusting the resistivity profile, but such an effect is out of scope of this course (anyway it is usually a weaker effect).

4. Under ideal Ohms law we have to lowest order (assuming one poloidal mode number),

$$\delta\psi = \frac{rB_0}{R_0}\left(\frac{n}{m} - \frac{1}{q}\right)\xi_0^r.$$

So, on the rational surface, if ξ_0^r isn't singular, $\delta\psi$ and hence δB^r will vanish on the rational surface. If $\xi_0^r \sim 1/x$, then $\delta\psi$ will be non-zero on the rational surface.

Finite radial magnetic field on the rational surface changes the topology of the magnetic field relative to the equilibrium field. Magnetic islands are generated (see course notes for this week). Yes, in practice, MHD modes require resistivity to change the topology. An ideal mode with singularity $\sim 1/x$ is academic, since it would be strongly stabilised. Hence any instability under the MHD model would have $\delta\psi = 0$. Under the Resistive MHD model, one can have non-zero $\delta\psi$ on the rational with ξ_0^r non-singular (though it will vary fast - see next questions) due to a balance with the extra diffusion term in,

$$\delta\psi - \frac{r\eta}{\gamma}\nabla^2\left(\frac{\delta\psi}{r}\right) = \frac{rB_0}{R_0}\left(\frac{n}{m} - \frac{1}{q}\right)\xi_0^r.$$

5. Solving numerically, and comparing with $1/z$ we have Fig. 1.

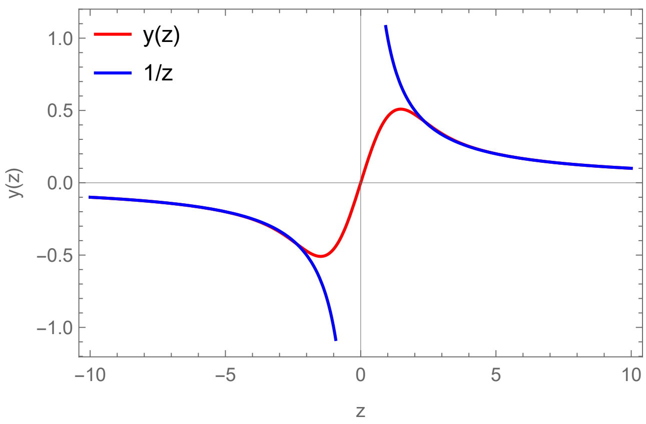

FIG. 1: Plot of numerical integral solution of $y(z)$ and comparison with $1/z$.

There is in fact an analytic solution to $y(z)$ which behaves well for the range of z where $y(z)$ disagrees with $1/z$.

It can be seen that its asymptote is indeed $y = 1/z$ for $|z| \gtrapprox 1$. And for $|z| \lessapprox 1$ the ideal case and the solution for $y(z)$ differ. The boundary at which the ideal and resistive solutions agree is thus at $Z \approx 1$, that is at

$$\frac{r - r_s}{\delta} \approx 1$$

where we note that $z = xr_s/\delta$ and $x = (r - r_s)/r_s$. The question defines the inertia-resistive layer width as the width bounding the range over which the resistive solution to $y(z)$ is different from the ideal-inertialess solution $1/z$. Hence the resistive layer with is

$$\approx 2\delta.$$

6. There are other ways to define the layer width as seen in the notes, e.g. we expect $(\xi^r)'' \sim \xi^r/\delta^2$. The estimate $\delta^2 \sim \xi^r/(\xi^r)''$ turns out to be roughly consistent with the approach shown in the previous question. Noting that $y(z)$ is proportional to $\xi^r(x)/\delta\psi(r_s)$, we have that in the layer (using $z = r_s x/\delta$),

$$\frac{(\xi^r)''}{\xi^r} \equiv \frac{1}{\xi^r(r)} \frac{d^2}{dr^2}\xi^r(r) = \frac{1}{r_s^2 \xi^r(x)} \frac{d^2}{dx^2}\xi^r(x) = \frac{1}{\delta^2 \xi^r(z)} \frac{d^2}{dz^2}\xi^r(z) = \frac{1}{\delta^2 y(z)} \frac{d^2}{dz^2} y(z).$$

Since, from the lecture slides,

$$\frac{d^2}{dz^2} y(z) = z^2 y(z) - z$$

then it is obvious that $y(z) \sim \delta^0\, d^2y(z)/dz^2$ over $|z| \lessapprox 1$. So that over the layer region we have that

$$\frac{(\xi^r)''}{\xi^r} = \frac{1}{\delta^2 y[z]} \frac{d^2}{dz^2} y(z) \sim \frac{1}{\delta^2}.$$

Hence the two means of identifying the resistive-layer width agree. Figure 2 plots that $y(z)$ and $d^2y(z)/dz^2$.

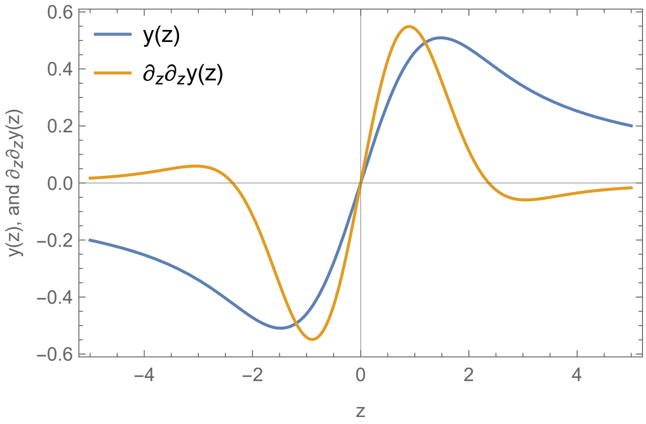

FIG. 2: Comparing $y(z)$ and $d^2y(z)/dz^2$.

7. Just manipulate each of the two equations and multiply them together to obtain,

$$\delta\,\Delta' = 2.12\frac{S^{1/2}}{ns}\left(\frac{\gamma}{\omega_A}\right)^{3/2}.$$

Hence, for $\delta\,\Delta' \ll 1$, we require,

$$2.12\frac{S^{1/2}}{ns}\left(\frac{\gamma}{\omega_A}\right)^{3/2} \ll 1$$

or

$$\frac{\gamma}{\omega_A} \ll \frac{(ns)^{2/3}}{S^{1/3}}.$$

If we take $ns = 1$, $S = 10^8$ we obtain,

$$\frac{\gamma}{\omega_A} \ll 2 * 10^{-3}.$$

Given that ideal instabilities grow typically with $\gamma/\omega_A \sim 10^{-2} - 10^{-3}$, and that we expect resistive instabilities to grow orders of magnitude more slowly, we can expect $\delta\,\Delta' \ll 1$ to hold (this is borne out in the next question).

8. Just requires eliminating the growth rate from the two equations given in the previous question, giving,

$$\frac{\delta}{r_s} \sim S^{-2/5}(ns)^{-2/5}$$

For the values taken in the previous question we obtain $\delta/r_s \sim 10^{-3}$. This of course means that $\delta\,\Delta' \sim 10^{-3}$, which indeed confirms $\delta\,\Delta' \ll 1$ and hence the constant-psi approximation.

For the growth rate, use the equation given in the previous question,

$$\frac{\gamma}{\omega_A} = \left[\frac{\Gamma(1/4)r_s\Delta'}{2\pi\Gamma(3/4)}\right]^{4/5} S^{-3/5}(ns)^{2/5}$$

and set $r_s\Delta' = 1$, $S = 10^8$ and $ns = 1$. Hence,

$$\frac{\gamma}{\omega_A} \sim S^{-3/5} \sim 10^{-5}$$

which is indeed much slower growth rate than ideal timescales.

9. This question is achieved via numerical integration. The result is shown in Fig. 3.

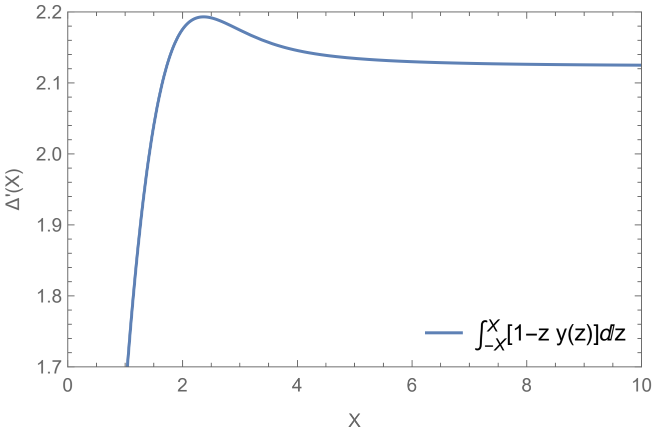

FIG. 3: Plot of Δ'_X showing saturation with $X > 1$.

The result provides validation of the matching approach taken, in particular matching the ideal region with the Δ' as X is pushed to infinity.

10. Just some algebra. Worth doing because the flux equation involving J_{phi} is the one you will see in the textbooks. The equation for ξ^r is less visible, but it is good to know the equations used in this course are solidly connected.