Plasma Instabilities

Solutions for Exercises Series 4

External kink modes and inertia treatment for ideal and resistive problems

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1. Indefinite integration gives,

$$\xi_0^r(x) = -\frac{\bar{\xi}_0}{\pi} \arctan\left(x\frac{ns_1\omega_A}{\gamma}\right) + C$$

The constant is obtained from the Dirichlet boundary condition at the edge, so that,

$$\xi_0^r(x) = \frac{\bar{\xi}_0}{2}\left[1 - \frac{2}{\pi}\arctan\left(x\frac{ns_1\omega_A}{\gamma}\right)\right]$$

and it is seen that $\xi_0^r(x=0) = \bar{\xi}_0/2$ as expected.

A plot similar to the one in the lecture notes. The layer width vanishes at marginal stability, the growth rate disappears, so it can't resolve the singularity. If the shear vanishes, the magnetic field line bending is zero everywhere, so the equations ar singular everywhere too, and in principle the inertia region fills the whole plasma.

2. For example, at half the layer width, we have approximately,

$$\xi_0^r(x-\delta/2) - \xi_{G0}^r(x-\delta/2) \approx \frac{3}{4}\bar{\xi}_0 - \bar{\xi}_0 = -\frac{1}{4}\bar{\xi}_0.$$

3. With $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel \boldsymbol{b}$ we have from $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$:

$$\boldsymbol{\nabla}\cdot(\xi_\parallel \boldsymbol{b}) = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp$$

And thus,

$$\boldsymbol{\nabla}\cdot\left(\xi_\parallel\frac{\boldsymbol{B}}{B}\right) = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp$$

Since $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$:

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\frac{\xi_\parallel}{B}\right) = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp.$$

Finally, using $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} = 0$ gives,

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\frac{\xi_\parallel}{B}\right) = -2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}.$$

4. Lowest order expansion is straightforward. The $\boldsymbol{\nabla}$ operator in κ requires that ϵ corrections are retained in $\boldsymbol{\nabla}B^2 \approx B_0^2\boldsymbol{\nabla}(1-2(r/R_0)\cos\theta$. Otherwise, we may take $\epsilon \to 0$, and we may take the leading order eigenfunctions

for $\boldsymbol{\xi}_\perp$. Hence, from the general definition of the magnetic operator from this series, we obtain the leading order operation,

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\frac{\boldsymbol{\xi}_\parallel}{B}\right)\approx\frac{1}{R_0}\left(\frac{\partial}{\partial\phi}+\frac{1}{q}\frac{\partial}{\partial\theta}\right)\xi_\parallel.$$

For the curvature, we note that we can drop the pressure (since $P\sim\epsilon^2B^2$), we may use $F=R_0B_0$ and also using $\boldsymbol{\nabla}B^2\approx B_0^2\boldsymbol{\nabla}(1-2(r/R_0)\cos\theta$, and noting that $\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}_\perp=\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}$, and thus $2\boldsymbol{\xi}\cdot\boldsymbol{\kappa}\approx-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}[(1-2(r/R_0)\cos\theta]$, we easily obtain the leading order expression:

$$2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\approx-\frac{2}{R_0}\left(\xi_0^r\cos\theta-\xi_{\perp0}^\theta\sin\theta\right).$$

Thus, the leading order magnetic equation is:

$$\left(\frac{\partial}{\partial\phi}+\frac{1}{q}\frac{\partial}{\partial\theta}\right)\xi_\parallel=-2\left(\xi_0^r\cos\theta-\xi_{\perp0}^\theta\sin\theta\right),$$

and furthermore that on setting

$$\xi_\parallel(r,\theta,\phi)=\hat{\xi}_\parallel(r,\theta)\exp(-im\theta+in\phi)$$

together with the same 'hat' notation for the poloidal and radial displacements, we easily obtain the following equation for $\hat{\xi}_\parallel$ is (by substituting this definition for $\xi_\parallel$ into the equation, and multiplying the equation by $R_0\exp(im\theta-in\phi)$:

$$\left[i\left(n-\frac{m}{q}\right)+\frac{1}{q}\frac{\partial}{\partial\theta}\right]\hat{\xi}_\parallel=-2\left(\hat{\xi}_0^r\cos\theta-\hat{\xi}_{\perp0}^\theta\sin\theta\right)$$

5. This can be demonstrated simply by substitution. We choose to verify the solution at the rational surface, so we replace m with nq. Notice that the solution of the equation has the general form

$$\hat{\xi}_\parallel=C-2q\left(\hat{\xi}_0^r\sin\theta+\hat{\xi}_{\perp0}^\theta\cos\theta\right)$$

where C is a constant of integration. This constant would introduce a flute term, in addition to the sideband terms:

$$\xi_\parallel=-\left\{C+q\left(\hat{\xi}_{\perp0}^\theta-i\hat{\xi}_0^r\right)\exp(i\theta)+q\left(\hat{\xi}_{\perp0}^\theta+i\hat{\xi}_0^r\right)\exp(-i\theta)\right\}\exp(-im\theta+in\phi)$$

As mentioned in the question, it can be shown by adding more physics, in particular intertia that the flute contribution C is zero to relevant order, and thus, as seen above, ξ comprises solely upper and lower poloidal sideband contributions,

$$\xi_\parallel=-q\left(\hat{\xi}_{\perp0}^\theta-i\hat{\xi}_0^r\right)\exp[-i(m-1)\theta+in\phi)]-q\left(\hat{\xi}_{\perp0}^\theta+i\hat{\xi}_0^r\right)\exp[-i(m+1)\theta+in\phi)]$$

which can be written succinctly as:

$$\xi_\parallel=-2q\left(\xi_0^r\sin\theta+\xi_{\perp0}^\theta\cos\theta\right).$$

Notice one can of course write the latter expression entirely in terms of ξ_0^r via the result obtained in an earlier question:

$$\xi_{\perp0}^\theta=-\frac{i}{m}\frac{\partial}{\partial r}\left(r\xi_0^r\right).$$

6. This problem requires evaluation of

$$\delta K_\parallel\propto\frac{1}{2\pi}\int_0^{2\pi}d\theta\left|\xi_\parallel\right|^2,$$

which should be compared with,

$$\delta K_\perp \propto \frac{1}{2\pi}\int_0^{2\pi} d\theta\, |\boldsymbol{\xi}_\perp|^2 \approx \int_0^{2\pi} d\theta \left[|\xi_0^r|^2 + \left|\xi_{\perp 0}^\theta\right|^2\right].$$

It is easiest to take the parallel displacement in the form

$$\xi_\parallel = -2q\left(\xi_0^r \sin\theta + \xi_{\perp 0}^\theta \cos\theta\right).$$

We also note that

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^2\theta = \frac{1}{2}, \quad \frac{1}{2\pi}\int_0^{2\pi} d\theta \sin^2\theta = \frac{1}{2}, \quad \frac{1}{2\pi}\int_0^{2\pi} d\theta \cos\theta\sin\theta = 0.$$

One then easily obtains that

$$\delta K_\parallel = 2q^2 \delta K_\perp.$$

For establishing the modification of the growth rate by inclusion of $\xi_\parallel$ and hence $\delta K_\parallel$, we obtain,

$$\begin{aligned}\delta K &= \delta K_\perp (1 + 2q^2) \\ &= -\omega^2 \frac{1}{2}\int d^3x\, \rho\, |\boldsymbol{\xi}_\perp|^2 (1 + 2q(x)^2) \\ &= -\omega^2 \frac{1}{2}\int d^3x\, \rho \left[|\xi^r|^2 + \left|\frac{1}{m}\frac{\partial(r\xi^r)}{\partial r}\right|^2\right](1 + 2q(x)^2)\end{aligned}$$

The dispersion relation that neglects parallel inertia is $\gamma^2 K_\perp = -\delta W$. In contrast, inclusion of the parallel displacement and hence inclusion of $\delta K_\parallel$, gives the dispersion relation $(1+2q^2)\gamma^2 K_\perp = -\delta W$. The growth rate $\gamma = -i\omega$ when including the parallel inertia, relative to the case without the parallel inertia, undergoes the replacement $\gamma \to \gamma/\sqrt{1+2q^2}$, where we note that unstable modes grown linearly as $\sim \exp(-i\omega t) = \exp(i\gamma t)$. Resistive instabilities in a torus also have growth rates renormalised by the same factor.

Hence codes or analytic theory which neglect the parallel displacement in the inertia predict growth rates that are larger by a factor $\sqrt{1+2q^2}$ compared to when solving full MHD. Such a model is sometimes called collisionless MHD (see Freidberg, *Ideal MHD*). Many codes and analytic treatments have alternative models for parallel dynamics, in particular they deploy alternatives to the equation of state. This is discussed briefly in the course, but it is quite complicated.

7. demonstration simply by substitution.

8. The important point to notice here is that the vector $\boldsymbol{n}$ perpendicular to the edge flux surface is identically in the $\boldsymbol{\nabla} r$ direction. In particular $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ and $\boldsymbol{B}\cdot\boldsymbol{\nabla} r = 0$, where $\boldsymbol{B}$ is the equilibrium field, which remains valid with Shafranov shifted and shaped flux surfaces. Clearly $r = a$ marks the edge minor radius, so in this question we may use the result for δB^r obtained in exercise week 3, and written in the question to lowest order in ϵ.

9. We have assumed that the wall geometry has roughly the same shape geometry as that of the last closed flux surface, so that in this simplified question, we may adopt a cylindrical coordinate system describing the radial and poloidal coordinates also in the vacuum. The poloidal coordinate is assumed to lie tangent to the wall, and the radial coordinate perpendicular to the wall (this approximation might not be very accurate for TCV!).

10. The boundary conditions (at $r = a$ and $r = b$ considered in the last two questions) yields the two constants. Rearranging gives the result in the convenient manner presented, i.e.

$$\delta\psi = \frac{B_0}{R_0}\left(\frac{n}{m} - \frac{1}{q_a}\right)\frac{\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m}{\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m}\, a\,\xi_a.$$

11. Recall from exercises series 3 that the toroidal, and parallel, perturbed fields are very small. Also, to lowest order we can neglect non-orthogonality, so $|\delta B|^2 \approx |\delta B^r|^2 + |\delta B^\theta|^2$. The result for the vacuum potential energy is obtained by substitution of the radial and poloidal fields in terms of the flux $\delta\psi$. Partial derivatives are

replaced by d/dr by the redefinition of the flux, as defined in the question. Note also that we use the lowest order screw pinch volume in the vacuum:

$$\begin{aligned}\int_V d^3x &= \int_a^b dr \int_0^{2\pi} d\phi\, R_0 \int_0^{2\pi} d\theta\, r \\ &= 4\pi^2 R_0 \int_a^b dr\, r.\end{aligned}$$

12. We take the result of the previous question and work as follows

$$\begin{aligned}\delta W_V &= 2\pi^2 R_0 \int_a^b dr\, r \left[\frac{m^2}{r^2}\delta\psi^2 + \left(\frac{d\delta\psi}{dr}\right)^2\right] \\ &= 2\pi^2 R_0 \int_a^b dr\, r \left[\frac{m^2}{r^2}\delta\psi^2 + \left(\frac{d\delta\psi}{dr}\right)\left(\frac{d\delta\psi}{dr}\right)\right] \\ &= 2\pi^2 R_0 \left\{\int_a^b dr\, r \left[\frac{m^2}{r^2}\delta\psi^2 - \frac{\delta\psi}{r}\left(r\frac{d\delta\psi}{dr}\right)\right]^{\nearrow 0} + r\left[r\delta\psi\frac{d\psi}{dr}\right]_a^b\right\} \\ &= 2\pi^2 R_0 \left[r\delta\psi\frac{d\psi}{dr}\right]_a^b\end{aligned}$$

The term that cancels appears via using integration by parts, and it cancels because of the identify given in the question (from vanishing current in the vacuum).

13. Simply by substitution, putting the boundary terms in δW_P together with the vacuum term, placing them all on the second line of the expression given in the question. It is necessary to evaluate $d\delta\psi/dr$, based on the derived expression for $\delta\psi$, and simplify $\delta\psi d\delta\psi/dr$.

14. Destabilising terms in δW are those that can be negative. The only term that can be negative is the first term on the second line, i.e.

$$\frac{2\pi^2 B_0^2}{R_0} a^2 \xi_0^r(a)^2 \frac{2}{q_a}\left(\frac{n}{m} - \frac{1}{q_a}\right).$$

Assuming that $q_a > 0$ and $n/m > 0$, this term can be destabilising if $q_a < m/n$, i.e. if there isn't an exact rational in the plasma. But q_a should be only slightly smaller than m/n, because if $q_a \ll m/n$ the magnetic field line bending contributions in the other terms, in all the plasma, will be strongly stabilising, outweighing the destabilising term. In contrast, of q_a is only slightly smaller than m/n we will have weak field line bending in the edge region of the plasma and also in the vacuum (since $|\delta B|^2$ in the vacuum will nearly vanish.

$b/a \to \infty$ is known as the no-wall limit. It is the most unstable case, as can be seen from λ, since it takes the smallest value ($\lambda = 1$ for $b/a \to \infty$) reducing the stabilising vacuum terms as much as possible. $b/a = 1$ is the case where the wall is placed directly on the plasma-vacuum interface. This case is absolutely stable, since the vacuum has been removed entirely. For this case the stabilising term in the second line of δW becomes infinitely larger than the destabilising term in the second line of δW, since $\lambda \to \infty$. Note however that for the case with no vacuum, $\xi_0^r(a) = 0$, so all surface and vacuum terms are in fact zero. The resulting problem at order ϵ^2 is stable.

The procedure for solving the stability problem is to vary the expression for δW with respect to ξ_0^r (Euler-Lagrange equation), from this, obtain $\xi_0^r(r)$ inside the plasma for a given q-profile, and substitute the solution to $\xi_0^r(r)$ back into δW. We call this δW_{min}. The sign of δW_{min} will determine if the particular mode m, n is stable or unstable for the choice of equilibrium q-profile. Nothing at this point has been determined about the growth rate (except whether the mode is growing or not).

15. As mentioned in the lecture notes, we recover normal mode equations by variation of the total energy $\delta K + \delta W$ with respect to $\boldsymbol{\xi}$ for constant ω. For the internal kink mode case it was seen that inclusion of the inertia strongly modifies the structure of the eigenfunction close to the rational surface. In particular it isn't correct to first obtain the minimum δW by variation of δW alone, and then calculated the growth rate via $\gamma^2 = -\delta W_{min}/K$, where $\gamma^2 K = \delta K$. The reason that approach goes wrong is that when the problem is done correctly, with

variation of the total energy, it is found that the structure of the eigenfunction yields that $K \sim (1/\gamma)(\xi^r)^2$, hence $\delta K \sim \gamma(\xi^r)^2$, and thus $\gamma \sim -\delta W_{min}$ (see earlier questions.

But, as hinted in the question, the external kink is a special case for which there is no rational surface inside the plasma for unstable cases ($q_a < m/n$). This means that there is no singularity in the Euler equation for ξ^r, and hence inertia will not have a dominant effect on the structure of the eigenfunction (unless the growth rate is very large - see how large below). In fact from the lecture notes we have that $\delta H = \delta K + \delta W$ where to relevant order in ϵ,

$$\begin{aligned}
\delta H =& \frac{2\pi^2 B_0^2}{R_0}\int_0^a dr\, r \left[\left(r\frac{d\xi_0^r}{dr}\right)^2 + \left(m^2-1\right)\left(\xi_0^r\right)^2\right]\left[\left(\frac{n}{m}-\frac{1}{q}\right)^2 + \frac{1}{m^2}\left(\frac{\gamma}{\omega_A}\right)^2\right] \\
&+ \frac{2\pi^2 B_0^2}{R_0} a^2 \xi_{r0}(a)^2 \left[\frac{2}{q_a}\left(\frac{n}{m}-\frac{1}{q_a}\right) + (1+m\lambda)\left(\frac{n}{m}-\frac{1}{q_a}\right)^2\right],
\end{aligned}$$

Providing q_a is not very close to m/n, we have that for $\gamma/\omega_A << 1$,

$$\left(\frac{n}{m}-\frac{1}{q}\right) \gg \left(\frac{\gamma}{\omega_A}\right)^2$$

in all the plasma. Variation of the energy δH with respect to ξ_0^r for constant γ can be undertaken approximately by setting $\gamma = 0$ in δH above. The obtained displacement can once again by substituted back into δW giving δW_{min}. It can also be substituted into K, giving an approximate growth rate $\gamma^2 = -\delta W_{min}/K$.

In the examples given in the lecture, we evaluated $\delta\hat{W}_{min}$ in terms of $\hat{\xi}^r(r) = \xi_0^r(r)/\xi_0^r(a)$. Let us now show how γ^2/ω_A^2 would be calculated with knowledge of $\delta\hat{W}_{min}$ and ξ_0^r, where,

$$\delta\hat{W} = \frac{R_0 \delta W_2}{2\pi^2 a^2 \xi_0^r(a)^2 B_0^2}.$$

Use

$$\delta K = -\gamma^2 \frac{1}{2}\int dx^3 \rho|\boldsymbol{\xi}|^2, \quad \rho = \frac{B_0^2}{R_0^2\omega_A^2}.$$

Adopting cylindrical geometry, we then obtain,

$$\delta K \approx -\frac{\gamma^2}{\omega_{A0}^2}\frac{2\pi a^2 B_0^2(\xi_0^r(a))^2}{R_0}\left(\frac{1}{a^2(\xi_0^r(a))^2}\int_0^a dr\,\frac{\rho}{\rho_0} r|\boldsymbol{\xi}|^2\right)$$

Also, from the variational problem discussed in the lectures, and also the calculation of the parallel flow given in these exercises, we have

$$\begin{aligned}
|\boldsymbol{\xi}|^2 &\approx (\xi_0^r)^2 + (\xi_0^\theta)^2 + (\xi_0^\phi)^2 \\
&= \left[(\xi_0^r)^2 + (\xi_0^\theta)^2\right](1+2q^2) \\
&= \left[(\xi_0^r)^2 + \left(\frac{1}{m}\frac{d}{dr}(r\xi_0^r)\right)^2\right](1+2q^2) \\
&= (\xi_0^r(a))^2\left[(\hat{\xi}^r)^2 + \frac{1}{m^2}\left(\frac{d}{dr}(r\hat{\xi}^r)\right)^2\right](1+2q^2).
\end{aligned}$$

Hence, we obtain,

$$\frac{\gamma^2}{\omega_{A0}^2} \approx -\frac{\delta\hat{W}_{min}(\hat{\xi}^r)}{\frac{1}{a^2}\int_0^a dr\, r\left[(\hat{\xi}^r(r))^2 + \frac{1}{m^2}\left(\frac{d}{dr}(r\hat{\xi}^r(r))\right)^2\right]\frac{\rho}{\rho_0}(1+2q(r)^2)}.$$

Examples of values for $\delta\hat{W}_{min}$ and profiles for $\hat{\xi}^r$ are given in the lecture notes for specific equilibria.

At least this analytic approach for calculating linear growth rates seem correct to me. I have only seen analytic growth rate calculations for external kink in the literature for special cases (e.g. in R. B. White *Theory of Fusion Plasmas* for a shear-free example). The calculation shown here would break down if growth rates become very large (external kink growth is fast!), or if the edge value of q is very close to the rational. The same approach could be used to non-resonant infernal modes and non-resonant internal kink modes I think. Needs checking as I haven't considered the approach before for non-resonant infernal modes and internal kink modes, so do comment.