Plasma Instabilities

Exercises Series 4

External kink modes and inertia treatment for ideal and resistive problems

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Questions on layer expansion theory: ideal internal kink mode

1. Using the expansion variable $x = (r - r_1)/r_1$ is was shown in the lecture that around the rational surface,

$$\frac{d\xi_0^r}{dx} = -\frac{\bar{\xi}_0}{\pi}\left(\frac{\gamma}{ns_1\omega_A}\right)\frac{1}{x^2 + \left(\frac{\gamma}{ns_1\omega_A}\right)^2}.$$

   Obtain from this $\xi_0^r(x)$ conforming to BC's

$$\lim_{x\to-\infty} \xi_0^r(x) = \bar{\xi}_0$$
$$\lim_{x\to\infty} \xi_0^r(x) = 0.$$

   Plot it over r, and mark on the plot the approximate characteristic width

$$\delta = \frac{\pi r_1 \gamma}{2ns_1\omega_A}.$$

   Comment on the layer width at marginal stability, and for vanishing magnetic shear.

2. Verify that the solution to $\xi_0^r(x)$ above, over $-\delta/r_1 < x < \delta/r_1$ has the property

$$\xi_0^r - \xi_{G0}^r \sim \epsilon^0 \bar{\xi}_0$$

   in the inner layer region, where

$$\xi_{G0}^r(r) = \bar{\xi}_0 H(r - r_1).$$

   This of course demonstrates that the ideal inertia corrections to ξ^r are order zero within a layer width δ of the rational surface.

Questions inertia enhancement due to parallel flow, valid for ideal and resistive MHD instabilities

3. As shown in week 3 lecture notes, near marginal stability modes are incompressible, i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$. For a given perpendicular displacement vector one may thus solve for the parallel displacement. Adopting $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel \boldsymbol{b}$ and using $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} = 0$ (from exercise series 3 and lecture notes) show that

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\left(\frac{\xi_\parallel}{B}\right) = 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}.$$

4. Continuing the previous question, and also the following results from exercise series 3:

$$\kappa = \frac{\boldsymbol{\nabla}_\perp}{B^2}\left(\frac{B^2}{2} + P\right),$$

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} = \frac{F}{R^2}\left[\frac{\partial}{\partial\phi} + \frac{1}{q}\frac{\partial}{\partial\theta}\right]$$

show that at lowest order in ϵ the equation satisfying $\xi_{\|}$ is:

$$\left(\frac{\partial}{\partial\phi}+\frac{1}{q}\frac{\partial}{\partial\theta}\right)\xi_{\|}=-2\left(\xi_0^r\cos\theta-\xi_{\perp 0}^{\theta}\sin\theta\right).$$

And furthermore that on setting

$$\xi_{\|}(r,\theta,\phi)=\hat{\xi}_{\|}(r,\theta)\exp(-im\theta+in\phi)$$

the equation for $\hat{\xi}_{\|}$ is:

$$\left[i\left(n-\frac{m}{q}\right)+\frac{1}{q}\frac{\partial}{\partial\theta}\right]\hat{\xi}_{\|}=-2\left(\hat{\xi}_0^r\cos\theta-\hat{\xi}_{\perp 0}^{\theta}\sin\theta\right)$$

5. Show that at the rational surface m/n the solution is:

$$\begin{aligned}\xi_{\|}&=-2q\left(\xi_0^r\sin\theta+\xi_{\perp 0}^{\theta}\cos\theta\right)\\&\equiv-q\left[\xi_{\perp 0}^{\theta}(\exp(i\theta)+\exp(-i\theta))-i\xi_0^r(\exp(i\theta)-\exp(-i\theta))\right].\end{aligned}$$

This can be undertaken by substitution. Note that the constant of integration has been assumed zero - this can be shown to be true by inclusion of more physics (inertia).

6. Near marginal stability $\xi_{\|}$ does not enter δW. It does not feature in the Force operator. But, it does still feature in the momentum equation (in the acceleration on the LHS) and thus in the inertia δK. This question investigates the modification toe δK via inclusion of ξ. Defining,

$$\delta K=\delta K_{\perp}+\delta K_{\|}$$

where

$$\delta K_{\perp}=-\omega^2\frac{1}{2}\int d^3x\,\rho\,|\boldsymbol{\xi}_{\perp}|^2$$

show that

$$\delta K_{\|}=2q^2\delta K_{\perp}$$

And therefore, show that inclusion of $\delta K_{\|}$ inside δK reduces the growth rate of a calculation absent of $\delta K_{\|}$ by $\gamma\to\gamma/\sqrt{1+2q^2}$.

This is important also because some codes assume incompressibility and otherwise neglect $\xi_{\|}$ (e.g. the TERPSICHORE code). Marginal stability conditions are therefore correct, but growth rates are overestimated relative to the standard MHD used in this course. There are various alternatives to the ideal MHD model, which is based on the equation of state. Models include collisionless MHD (see Freidberg, *Ideal MHD*), and kinetic MHD.

Questions on ideal external kink modes

7. The radial perturbed magnetic field in terms of the flux $\delta\psi$ is $\delta B^r=im\delta\psi/r$. The equation for the perturbed flux in a vacuum obeys (to leading order):

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\delta\psi}{dr}\right)-\frac{m^2}{r^2}\delta\psi=0 \tag{1}$$

Verify by substitution that the general solution in the vacuum is

$$\delta\psi=\alpha r^m+\beta r^{-m}. \tag{2}$$

8. As shown in exercise series 3, the lowest order radial magnetic field in the plasma is,

$$\delta B^r = \frac{iB_0}{R_0 q}\left[nq - m\right]\xi_0^r.$$

Using again $\delta B^r = im\delta\psi/r$ (which applies in the plasma too), use the boundary condition at the plasma-vacuum interface (at $r = a$):

$$\boldsymbol{n}\cdot\boldsymbol{\delta}\hat{\boldsymbol{B}}|_a = \boldsymbol{n}\cdot\boldsymbol{\nabla}\times(\boldsymbol{\xi}_\perp\times\boldsymbol{B})|_a$$

to yield that

$$\delta\psi_a = \delta\psi(r=a) = \frac{B_0}{R_0}\left(\frac{n}{m} - \frac{1}{q_a}\right)a\,\xi_a \tag{3}$$

where $\xi_a = \xi_0^r(a)$.

9. What have we assumed about the geometry of the wall if the general boundary condition at the wall,

$$\boldsymbol{n}\cdot\boldsymbol{\delta}\hat{\boldsymbol{B}}|_{wall} = 0$$

yields

$$\delta\psi_b = \delta\psi(r=b) = 0? \tag{4}$$

10. Use the BC's of Eqs. (3) and (4) to obtain the constants α and β, and show that the vacuum flux is

$$\delta\psi = \frac{B_0}{R_0}\left(\frac{n}{m} - \frac{1}{q_a}\right)\frac{\left(\frac{r}{b}\right)^m - \left(\frac{b}{r}\right)^m}{\left(\frac{a}{b}\right)^m - \left(\frac{b}{a}\right)^m}\,a\,\xi_a. \tag{5}$$

11. Consider the vacuum potential energy,

$$\delta W_V = \frac{1}{2}\int_V d^3x\,|\boldsymbol{\delta B}|^2.$$

Use that the poloidal magnetic field in the vacuum is $\delta B^\theta = \partial\delta\psi/\partial r$, and $|\delta B|^2 \approx |\delta B^r|^2 + |\delta B^\theta|^2$ (see exercise series 3) to yield that,

$$\delta W_V \approx 2\pi^2 R_0\int_a^b dr\,r\left[\frac{m^2}{r^2}\delta\psi^2 + \left(\frac{d\delta\psi}{dr}\right)^2\right]$$

where we have assumed replacement $\delta\psi \to \delta\psi/\exp(in\phi - im\theta - i\omega t)$.

12. Continuing the last question, use integration by parts and Eq. (1) to give

$$\delta W_V \approx 2\pi^2 R_0\left[r\delta\psi\frac{d\delta\psi}{dr}\right]_a^b. \tag{6}$$

13. The total potential energy is $\delta W = \delta W_P + \delta W_V$, where to relevant order (as shown in the lecture notes)

$$\begin{aligned}\delta W_P &= \frac{2\pi^2 B_0^2}{R_0}\left\{\int_0^a dr\,r\left[\left(r\frac{d\xi_0^r}{dr}\right)^2 + \left(m^2-1\right)(\xi_0^r)^2\right]\left(\frac{n}{m} - \frac{1}{q}\right)^2 + \right.\\ &\left. a^2(\xi_0^r(a))^2\left[\frac{2}{q_a}\left(\frac{n}{m} - \frac{1}{q_a}\right) + \left(\frac{n}{m} - \frac{1}{q_a}\right)^2\right]\right\}\end{aligned} \tag{7}$$

with $q_a = q(a)$. By inserting the vacuum solution of Eq. (5) into Eq. (6) show that the total potential energy is:

$$\begin{aligned}\delta W &= \frac{2\pi^2 B_0^2}{R_0}\left\{\int_0^a dr\,r\left[\left(r\frac{d\xi_0^r}{dr}\right)^2 + \left(m^2-1\right)^2(\xi_0^r)^2\right]\left(\frac{n}{m} - \frac{1}{q}\right)^2\right.\\ &\left.+a^2\xi_0^r(a)^2\left[\frac{2}{q_a}\left(\frac{n}{m} - \frac{1}{q_a}\right) + (1+m\lambda)\left(\frac{n}{m} - \frac{1}{q_a}\right)^2\right]\right\},\end{aligned} \tag{8}$$

where

$$\lambda = \frac{1 + (a/b)^{2m}}{1 - (a/b)^{2m}}.$$

14. Which term in δW can be destabilising? What is the condition for q_a in order for that term to be destabilising? What are the effects of and physical meanings of the limits $b/a \to \infty$ and $b/a \to 1$. How does one establish from δW above whether the plasma is unstable to an external kink displacement. What is the procedure?

15. How can the growth rate of an external kink be calculated? Hint: as seen, there is no rational surface inside the plasma for unstable modes for which $q_a < m/n$, so the inclusion of inertia (when performing variation of the total energy $\delta K + \delta W$) does not correct the structure of the eigenfunction significantly relative to the case where $\boldsymbol{\xi}$ is calculated by minimisation of just δW.