Plasma Instabilities

Solutions for Exercises Series 3

Theory of linear ideal MHD

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1. First we have

$$\begin{aligned}\boldsymbol{\kappa} = (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b} &= \left(\frac{\boldsymbol{B}}{B}\cdot\boldsymbol{\nabla}\right)\frac{\boldsymbol{B}}{B} \\ &= \frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}}{B^2} + \frac{\boldsymbol{B}}{B}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\left(\frac{1}{B}\right) \\ &= \frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}}{B^2} - \frac{\boldsymbol{B}}{B^3}(\boldsymbol{B}\cdot\boldsymbol{\nabla})B \\ &= \frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}}{B^2} - \frac{\boldsymbol{B}}{B^4}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\left(\frac{B^2}{2}\right).\end{aligned} \tag{1}$$

Now consider

$$\begin{aligned}\frac{1}{B^2}\boldsymbol{\nabla}_\perp\left(\frac{B^2}{2}+P\right) &= \frac{1}{B^2}\left(\boldsymbol{\nabla}-\boldsymbol{b}(\boldsymbol{b}\cdot\boldsymbol{\nabla})\right)\left(\frac{B^2}{2}\right)+\frac{1}{B^2}\boldsymbol{\nabla}P(r) \\ &= \frac{1}{B^2}\boldsymbol{\nabla}\left(\frac{B^2}{2}\right) - \frac{\boldsymbol{B}}{B^4}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\left(\frac{B^2}{2}\right) + \frac{1}{B^2}\boldsymbol{J}\times\boldsymbol{B} \\ &= \frac{1}{B^2}\boldsymbol{\nabla}\left(\frac{B^2}{2}\right) - \frac{\boldsymbol{B}}{B^4}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\left(\frac{B^2}{2}\right) - \frac{1}{B^2}\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B})\end{aligned}$$

where we have used Force balance and Amperes law, as suggested in the question, and that $P = P(r)$ so that $(\boldsymbol{b}\cdot\boldsymbol{\nabla})P = 0$, i.e. P is constant on a flux surface (thus does not vary along the field lines).

Now use vector identity,

$$\boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \frac{1}{2}\boldsymbol{\nabla}B^2 - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}$$

to yield

$$\begin{aligned}\frac{1}{B^2}\boldsymbol{\nabla}_\perp\left(\frac{B^2}{2}+P\right) &= \frac{1}{B^2}\boldsymbol{\nabla}\left(\frac{B^2}{2}\right) - \frac{\boldsymbol{B}}{B^4}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\left(\frac{B^2}{2}\right) - \frac{1}{B^2}\left(\frac{1}{2}\boldsymbol{\nabla}B^2 - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}\right) \\ &= \frac{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}}{B^2} - \frac{\boldsymbol{B}}{B^4}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\left(\frac{B^2}{2}\right).\end{aligned} \tag{2}$$

We see that Eqs. (1) is identical to (2), so it follows that,

$$\kappa = \frac{\boldsymbol{\nabla}_\perp}{B^2}\left(\frac{B^2}{2}+P\right).$$

2. First, consider $2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}$, which from the result of the previous question, we have

$$2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} = \frac{1}{B^2}\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}(B^2+2P).$$

Recalling the ordering of the pressure mentioned in the question, that $B^2/B_0^2 = (R_0^2/R^2)\left(1+O(\epsilon^2)\right)$, and also noting that Jacobian $\mathcal{J}_\theta = rR^2/R_0$, we have that,

$$2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} = \frac{\mathcal{J}_\theta}{r}\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\left(\frac{r}{\mathcal{J}_\theta}\right)\left(1+O(\epsilon^2)\right).$$

And therefore,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} &= \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \frac{\mathcal{J}_\theta}{r}\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(\frac{r}{\mathcal{J}_\theta}\right)\left(1+O(\epsilon^2)\right)\\ &= \frac{\mathcal{J}_\theta}{r}\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right)\left(1+O(\epsilon^2)\right).\end{aligned}$$

Consider now $\boldsymbol{\nabla}\cdot(X\boldsymbol{\xi}_\perp)$, which can be written as

$$\boldsymbol{\nabla}\cdot(X\boldsymbol{\xi}_\perp) = \frac{1}{\mathcal{J}_\theta}\left[\frac{\partial}{\partial r}\left(\mathcal{J}_\theta X\xi^r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\mathcal{J}_\theta X\xi_\perp^\theta\right) + \frac{1}{R}\frac{\partial}{\partial\phi}\left(\mathcal{J}_\theta X\xi_\perp^\phi\right)\right],$$

where

$$\xi^r = \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}r, \quad \xi_\perp^\theta = r\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\theta, \quad \xi_\perp^\phi = R\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\phi.$$

Hence, we see that,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} &= \frac{\mathcal{J}_\theta}{r}\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right)\left(1+O(\epsilon^2)\right) &(3)\\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\xi^r\right) + \frac{\partial}{\partial\theta}\left(\xi_\perp^\theta\right) + \frac{r}{R}\frac{\partial}{\partial\phi}\left(\xi_\perp^\phi\right)\right](1+O(\epsilon^2)). &(4)\end{aligned}$$

3. Form the dot product of

$$\boldsymbol{B} = F(r)\boldsymbol{\nabla}\phi + \psi'\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}r$$

with

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}\phi\frac{\partial}{\partial\phi} + \boldsymbol{\nabla}\theta\frac{\partial}{\partial\theta} + \boldsymbol{\nabla}r\frac{\partial}{\partial r}$$

to give,

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} = \frac{F}{R^2}\frac{\partial}{\partial\phi} + \frac{\psi'}{\mathcal{J}_\theta}\frac{\partial}{\partial\theta}$$

Here we have used $\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi = 1/R^2$ and $(\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}r)\cdot\boldsymbol{\nabla}\theta = 1/\mathcal{J}_\theta$.

Then use from the second lecture

$$\psi' = \frac{r}{R_0}\frac{F(r)}{q(r)}$$

and

$$\mathcal{J}_\theta = \frac{rR(r,\theta)^2}{R_0}$$

to yield the demanded result

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} = \frac{F}{R^2}\left[\frac{\partial}{\partial\phi} + \frac{1}{q}\frac{\partial}{\partial\theta}\right].$$

The magnetic operator is very important. It measures the variation of quantities along the magnetic field lines. When operating on a fluctuation it pulls out the **parallel wave vector** $k_\parallel$. Since $\boldsymbol{b} = \boldsymbol{B}/B$, we simply divide $\boldsymbol{B} \cdot \boldsymbol{\nabla}$ by B to obtain:

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial l} = \frac{F}{BR^2}\left[\frac{\partial}{\partial \phi} + \frac{1}{q}\frac{\partial}{\partial \theta}\right].$$

And, since we have normal modes of type $\exp(i\boldsymbol{k} \cdot \boldsymbol{x})$, then $\boldsymbol{b} \cdot \boldsymbol{\nabla} = ik_\parallel$, thus

$$k_\parallel = -i\boldsymbol{b} \cdot \boldsymbol{\nabla} = -i\frac{F}{BR^2}\left[\frac{\partial}{\partial \phi} + \frac{1}{q}\frac{\partial}{\partial \theta}\right].$$

4. First, it is useful to check that $\xi_\perp^\phi = \xi_{\perp\phi}$. This is easily tackled by forming the dot product of the covariant form of $\boldsymbol{\xi} = \xi_r \boldsymbol{\nabla} r + r\xi_{\perp\theta}\boldsymbol{\nabla}\theta + R\xi_{\perp\phi}\boldsymbol{\nabla}\phi$ with $= \boldsymbol{\nabla}\phi$,

$$\xi_\perp^\phi \equiv R\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\phi = R\left(\xi_r \boldsymbol{\nabla} r + r\xi_{\perp\theta}\boldsymbol{\nabla}\theta + R\xi_{\perp\phi}\boldsymbol{\nabla}\phi\right) \cdot \boldsymbol{\nabla}\phi$$

and using $\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi = 1/R^2$, clearly gives

$$\xi_\perp^\phi = \xi_{\perp\phi}.$$

We can obtain $\xi_\perp^\phi$ in terms of $\xi_{\perp\theta}$ via the definition of $\boldsymbol{B} = F(r)\boldsymbol{\nabla}\phi + \psi'\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r$. In particular we use,

$$R\boldsymbol{\nabla\phi} = \frac{R}{F}\boldsymbol{B} - \frac{\psi' R}{F}\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r.$$

We then obtain,

$$\xi_\perp^\phi \equiv R\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\phi = R\boldsymbol{\xi}_\perp \cdot \left(\frac{R}{F}\boldsymbol{B} - \frac{\psi' R}{F}\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r\right).$$

Noting that $\boldsymbol{\xi}_\perp \cdot \boldsymbol{B} = 0$ then gives,

$$\xi_\perp^\phi = R\left(\xi_r \boldsymbol{\nabla} r + r\xi_{\perp\theta}\boldsymbol{\nabla}\theta + R\xi_{\perp\phi}\boldsymbol{\nabla}\phi\right) \cdot \left(-\frac{\psi' R}{F}\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r\right).$$

As in the last question use $(\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla} r) \cdot \boldsymbol{\nabla}\theta = 1/\mathcal{J}_\theta$, together with (from the lecture notes)

$$\psi' = \frac{r}{R_0}\frac{F(r)}{q(r)}$$

and

$$\mathcal{J}_\theta = \frac{rR(r,\theta)^2}{R_0}.$$

Hence,

$$\xi_\perp^\phi = \frac{r}{qR}\xi_{\perp\theta}$$

and since we have found $\xi_\perp^\phi = \xi_{\perp\phi}$, then

$$\xi_{\perp\phi} = \frac{r}{qR}\xi_{\perp\theta}$$

5. For the poloidal contravariant component we have that

$$\xi_\perp^\theta \equiv r\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\theta = r\left(\xi_r \boldsymbol{\nabla} r + r\xi_{\perp\theta}\boldsymbol{\nabla}\theta + R\xi_{\perp\phi}\boldsymbol{\nabla}\phi\right) \cdot \boldsymbol{\nabla}\theta = \xi_{\perp\theta}(1 + O(\epsilon))$$

where as pointed out in the question, the order ϵ correction is related to non-orthogonality associated with $\boldsymbol{\nabla} r \cdot \boldsymbol{\nabla}\theta$ (this can be shown rigourously by evaluating the metric tensor for straight field line coordinates (lecture 6 onwards). Hence, from the results of the previous question, in particular $\xi_{\perp}^{\phi} = (r/(qR))\xi_{\perp\theta}$ we have that

$$\xi_{\perp}^{\phi} = \xi_{\perp}^{\theta}\frac{\epsilon}{q}(1+O(\epsilon)).$$

Continuing the question, we use the solution to question 2,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\xi^{r}) + \frac{\partial}{\partial\theta}\left(\xi_{\perp}^{\theta}\right) + \frac{r}{R}\frac{\partial}{\partial\phi}\left(\xi_{\perp}^{\phi}\right)\right](1+O(\epsilon^{2}))$$

Using now that $\xi_{\perp}^{\phi} \sim \epsilon\xi_{\perp}^{\theta}$ and $\partial/\partial\phi \sim -q^{-1}\partial/\partial\theta$ we find that

$$\frac{\partial}{\partial\theta}\left(\xi_{\perp}^{\theta}\right) + \frac{r}{R}\frac{\partial}{\partial\phi}\left(\xi_{\perp}^{\phi}\right) = \frac{\partial}{\partial\theta}\left(\xi_{\perp}^{\theta}\right)(1+O(\epsilon^{2}q^{0}))$$

Hence, we easily obtain the required result,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_{\perp} + 2\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\kappa} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\xi^{r}) + \frac{\partial}{\partial\theta}\left(\xi_{\perp}^{\theta}\right)\right](1+O(\epsilon^{2})).$$

In the lecture course we will this drop the $O(\epsilon^2)$ corrections in the ϵ expansion of δW up to δW_2, since the ϵ^2 corrections would appear at δW_4.

6. This problem is easily tackled since the equations mentioned in the question give

$$\frac{\partial}{\partial r}(r\xi_{0}^{r}) + \frac{\partial}{\partial\theta}\left(\xi_{\perp 0}^{\theta}\right) = 0,$$

which of course forces $\delta W_0 = 0$. Hence with $\boldsymbol{\xi}_{\perp 0}(r,\theta,\phi) = \hat{\boldsymbol{\xi}}_{\perp 0}(r)\exp(-im\theta + in\phi)$:

$$\xi_{\perp 0}^{\theta} = -\frac{i}{m}\frac{\partial}{\partial r}(r\xi_{0}^{r}).$$

(in fact, we can easily see that

$$\frac{\partial}{\partial r}(r\xi_{1}^{r}) + \frac{\partial}{\partial\theta}\left(\xi_{\perp 1}^{\theta}\right) = 0,$$

holds for $\boldsymbol{\xi}_{\perp 1}$ too, which minimises δW_2. But $\boldsymbol{\xi}_{\perp 1}$ has more than one poloidal mode number in a torus, so $\xi_{\perp 1}^{\theta} \neq -(i/m)\partial/\partial r(r\xi_{1}^{r})$).

7. We start from the identity

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{\nabla}\times\boldsymbol{B}$$

so that

$$\boldsymbol{\nabla}\cdot[(\boldsymbol{\xi}_{\perp}\times\boldsymbol{B})\times\boldsymbol{\nabla}X] = \boldsymbol{\nabla}X\cdot\boldsymbol{\nabla}\times(\boldsymbol{\xi}_{\perp}\times\boldsymbol{B}) - (\boldsymbol{\xi}_{\perp}\times\boldsymbol{B})\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}X).$$

Using $\boldsymbol{\nabla}\times\boldsymbol{\nabla}X = 0$ for any X, we clearly have the desired result,

$$\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}X = \boldsymbol{\nabla}\cdot[(\boldsymbol{\xi}_{\perp}\times\boldsymbol{B})\times\boldsymbol{\nabla}X],$$

(which is the intermediary result) where

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}_{\perp}\times\boldsymbol{B}).$$

We now use the triple product rule to give,

$$\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}X = -\boldsymbol{\nabla}\cdot[\boldsymbol{\xi}_{\perp}(\boldsymbol{B}\cdot\boldsymbol{\nabla}X) - \boldsymbol{B}(\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\nabla}X)]$$

And finally, and using $\boldsymbol{\nabla}(f\boldsymbol{A}) = f\boldsymbol{\nabla}\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}f$, and noting that $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, obtains the desired result,

$$\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}X = (\boldsymbol{B}\cdot\boldsymbol{\nabla})(\boldsymbol{\xi}_{\perp}\cdot\boldsymbol{\nabla}X) - \boldsymbol{\nabla}\cdot[\boldsymbol{\xi}_{\perp}(\boldsymbol{B}\cdot\boldsymbol{\nabla}X)] \tag{5}$$

8. Noting that $\boldsymbol{B}\cdot\boldsymbol{\nabla}r=0$ (no equilibrium field across flux surfaces) we have from Eq. (5):

$$\delta B^r=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi^r.$$

For the second part of the question, we use

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}=\frac{F}{R^2}\left[\frac{\partial}{\partial\phi}+\frac{1}{q}\frac{\partial}{\partial\theta}\right].$$

Also, look in the lecture notes and verify that to lowest order in ϵ we have that $F=R_0B_0$, $R=R_0$ and the displacement is of the form

$$\xi_0^r(r,\theta,\phi)=\hat{\xi}_0^r(r)\exp(-im\theta+in\phi),$$

i.e. that a single poloidal harmonic can be identified to lowest order in ϵ (as in a the cylindrical approximation $(r,\theta,z=R_0\phi)$ of the torus). One then easily finds that

$$\delta B_0^r=\frac{iB_0}{R_0q}\left[nq-m\right]\xi_0^r.$$

Clearly δB_0^r vanishes in a rational surface. This prevents a change of topology. A change of topology is only possible if we have some dissipation, e.g. resistivity. The island structure of a tearing mode is associated with non-zero δB_0^r on a rational surface.

9. We have from Eq. (5) that

$$\delta B^\theta\equiv r\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}\theta=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi_\perp^\theta-r\boldsymbol{\nabla}\cdot\left[\boldsymbol{\xi}_\perp(\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta)\right].$$

where $\xi_\perp^\theta=r\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\theta$. Use now that

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\theta=\frac{\psi'}{\mathcal{J}_\theta}=\frac{rF}{qR_0\mathcal{J}_\theta}$$

to give,

$$\delta B^\theta=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi_\perp^\theta-\frac{rF}{qR_0}\boldsymbol{\nabla}\cdot\left[\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right]-\frac{r^2}{R_0\mathcal{J}_\theta}\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(\frac{F}{q}\right)$$

Noting that $q=q(r)$ and $F=F(r)$, and that $\mathcal{J}_\theta=rR^2/R_0$ we obtain the first desired result,

$$\delta B^\theta=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi_\perp^\theta-\frac{rF}{qR_0}\boldsymbol{\nabla}\cdot\left(\frac{r\boldsymbol{\xi}_\perp}{\mathcal{J}_\theta}\right)-\frac{r}{R^2}\xi^r\frac{d}{dr}\left(\frac{F}{q}\right).$$

For further reduction, we can use that $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp+2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}=0$ which applies to leading order and first order displacements. As a result, from Eq. (3), together with $F=B_0R_0(1+O(\epsilon^2))$ we have,

$$\delta B^\theta=\left\{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi_\perp^\theta-\frac{rR_0B_0}{R^2}\xi^r\frac{d}{dr}\left(\frac{1}{q}\right)\right\}(1+O(\epsilon^2)).$$

we define the magnetic shear

$$s=\frac{r}{q}\frac{dq}{dr},$$

so that,

$$\delta B^\theta=\left\{(\boldsymbol{B}\cdot\boldsymbol{\nabla})\xi_\perp^\theta+\frac{R_0B_0s(r)\xi^r}{q(r)R^2}\right\}(1+(\epsilon^2)).$$

Adopting the lowest order displacement, which is appropriate for δB_0^θ, we thus easily have the lowest order poloidal perturbed field,

$$\delta B_0^\theta=\frac{iB_0\xi_0^\theta}{R_0q}\left[nq-m\right]+\frac{B_0\xi_0^r}{R_0q}s(r).$$

Note this can of course be written entirely in terms of the radial displacement. From the previous question,

$$\xi^\theta_{\perp 0} = -\frac{i}{m}\frac{\partial}{\partial r}\left(r\xi^r_0\right).$$

we have that finally,

$$\delta B^\theta_0 = \frac{B_0}{R_0 q(r)}\left[s(r)\xi^r_0 + \frac{nq(r)-m}{m}\frac{\partial}{\partial r}\left(r\xi^r_0\right)\right].$$

10. Since the variation of F is weak, and $\xi^\phi_\perp \sim \epsilon\xi^\theta_\perp$, we my ignore δB^ϕ in the leading order construction of $\delta B_\perp$. Neglecting non-orthogonality corrections, which introduce higher order ϵ terms, we have

$$|\delta B_{\perp 0}|^2 = |\delta B^r_0|^2 + \left|\delta B^\theta_0\right|^2 = \left(\frac{B_0}{R_0 q}\right)^2\left[(nq(r)-m)^2\left(\xi^r_0\right)^2 + \left\{\frac{nq(r)-m}{m}\frac{\partial}{\partial r}\left(r\xi^r_0\right) + s(r)\xi^r_0\right\}^2\right]$$

Instabilities align with rational surfaces because instabilities occur where the stabilising field line bending energy is minimised, and this clearly occurs where $q(r) = m/n$ (consider δB^r_0 and δB^θ_0 from the previous questions). As we saw at the start of this exercise, the parallel wavenumber $k_\parallel = \partial/\partial l = \boldsymbol{b}\cdot\boldsymbol{\nabla}$. For leading order radial and poloidal displacement fluctuations this is of course zero on the rational surface.

But it is clear that $\delta B^2_{\perp 0}$ is non-zero even on a rational surface if the magnetic shear is not zero. Tokamaks require magnetic shear in order to achieve stable operation. Sometimes tokamaks operate with a q-profile that has a local minimum, i.e. a location where $s = 0$ but $q'' \neq 0$. The location of this minimum must be chosen to avoid low order rational surfaces, that is a rational surface where n and m are both small (these are long wavelength modes).

11. Start with, as usual,

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times\left(\boldsymbol{\xi}_\perp\times\boldsymbol{B}\right)$$

Dotting with $\boldsymbol{b}$ and using the identity,

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{A}\times\boldsymbol{B}\right) = \boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{\nabla}\times\boldsymbol{B}$$

we have using Amperes Law and force balance:

$$\begin{aligned}
\delta B_\parallel &= \frac{\boldsymbol{B}}{B}\cdot\boldsymbol{\nabla}\times\left(\boldsymbol{\xi}_\perp\times\boldsymbol{B}\right) = \\
&= \frac{1}{B}\left\{\boldsymbol{\nabla}\cdot\left[\left(\boldsymbol{\xi}_\perp\times\boldsymbol{B}\right)\times\boldsymbol{B}\right] + \left(\boldsymbol{\xi}_\perp\times\boldsymbol{B}\right)\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\right\} \\
&= \frac{1}{B}\left\{\boldsymbol{\nabla}\cdot\left[\cancelto{0}{\boldsymbol{B}\boldsymbol{\xi}_\perp\cdot\boldsymbol{B}} - B^2\boldsymbol{\xi}_\perp\right] + \left(\boldsymbol{\xi}_\perp\times\boldsymbol{B}\right)\cdot\left(\boldsymbol{\nabla}_\perp\times\boldsymbol{B}\right)\right\} \\
&= -B\left[\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \frac{1}{B^2}\xi_\perp\cdot\boldsymbol{\nabla}B^2\right] + \frac{\left(\boldsymbol{\xi}_\perp\times\boldsymbol{B}\right)\cdot\boldsymbol{J}}{B} \\
&= -B\left[\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \frac{2}{B^2}\xi_\perp\cdot\boldsymbol{\nabla}\left(\frac{B^2}{2} + P\right) - \frac{2}{B^2}\xi_\perp\cdot\boldsymbol{\nabla}P\right] - \frac{\boldsymbol{\xi}_\perp\cdot\left(\boldsymbol{J}\times\boldsymbol{B}\right)}{B} \\
&= -B\left[\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} - \frac{2}{B^2}\xi_\perp\cdot\boldsymbol{\nabla}P\right] - \frac{\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}P}{B} \\
&= -B\left[\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right] + \frac{\xi^r}{B}\frac{dP}{dr}.
\end{aligned}$$

We have seen in this exercise sheet that instabilities occur for

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa} = 0,$$

so that corresponding parallel magnetic fluctuations are thus:

$$\delta B_\parallel \approx \frac{\xi^r}{B}\frac{dP}{dr}.$$

It is well known that finite $\delta B_{\parallel}$ effects are associated with finite beta effects (or beta-gradient effects). This can be seen clearly in the answer to this question. For that reason codes that neglect $\delta B_{\parallel}$ effects (notably some MHD codes, and some gyrokinetic codes) effectively neglect finite β effects. It can be rigourously shown (Graves, PPCF 2019) that neglecting $\delta B_{\parallel}$ introduces an artificial stabilising effect which can be important for some pressure gradient driven instabilities (such as interchange and internal kink modes). It is for this reason that some codes have not been able to obtain internal kink modes in a torus (e.g. JOREK MHD code, and GTC gyrokinetic code).