Plasma Instabilities

Exercises Series 2

Grad Shafranov Equation

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1. From Eqs. (1.15) and (1.22) in the lecture notes, show that the Grad-Shafranov equation can be written as

$$\Delta^* A + \Delta^* B = -R^2 \frac{dP}{dr} - F\frac{dF}{dr}$$

where (using the notation developed in the second lecture, notably Jacobian $\mathcal{J}_\omega$, and ε is a tag),

$$\Delta^* A = (\psi')^2 \frac{R^2}{\mathcal{J}_\omega}\left[\frac{\partial}{\partial r}\left(\frac{g_{\omega,\omega}}{\mathcal{J}_\omega}\right) - \frac{\partial}{\partial \omega}\left(\frac{g_{r,\omega}}{\mathcal{J}_\omega}\right)\right], \quad \Delta^* B = \psi'\,\psi''\frac{R^2}{\mathcal{J}_\omega^2} g_{\omega,\omega}.$$

2. Taking (see question 12)

$$\begin{aligned}
\mathcal{J}_\omega &= rR_0\left\{1 + \varepsilon(\epsilon - \Delta')\cos\omega + \varepsilon\sum_{m=2}^{\infty}\left(S_m' - (m-1)\frac{S_m}{r}\right)\cos(m\omega) + O(\varepsilon^2)\right\} \\
g_{r,\omega} &= r\left\{\varepsilon\Delta'\sin\omega - \varepsilon\sum_{m=2}^{\infty}\left(S_m' + (m-1)\frac{S_m}{r}\right)\sin(m\omega) + O(\varepsilon^2)\right\} \\
g_{\omega,\omega} &= r^2\left\{1 - \varepsilon 2\sum_{m=2}^{\infty}(m-1)\frac{S_m}{r}\cos(m\omega) + O(\varepsilon^2)\right\},
\end{aligned}$$

show that

$$\Delta^* A = \frac{\psi'^2}{r}\left\{1 + \varepsilon\left(r\Delta'' + \Delta' - \frac{r}{R_0}\right)\cos\omega - \varepsilon\sum_{m=2}^{\infty}\left(rS_m'' + S_m' + (1-m^2)\frac{S_m}{r}\right)\cos(m\omega) + O(\varepsilon^2)\right\}$$

and

$$\Delta^* B = \psi'\,\psi''\left\{1 + \varepsilon 2\Delta'\cos\omega - \varepsilon 2\sum_{m=2}^{\infty}S_m'\cos(m\omega) + O(\varepsilon^2)\right\}.$$

3. Show the following equations can be identified from the Grad Shafranov equation, defined in question 1:

$$\begin{aligned}
&\frac{1}{2r^2}\left(r^2\psi'^2\right)' + R_0^2 P' + FF' = 0, \\
&\Delta'' + \left(2\frac{\psi''}{\psi'} + \frac{1}{r}\right)\Delta' - \frac{1}{R_0} + 2\frac{rR_0P'}{(\psi')^2} = 0, \\
&S_m'' + \left(2\frac{\psi''}{\psi'} + \frac{1}{r}\right)S_m' + \frac{1-m^2}{r^2}S_m = 0.
\end{aligned}$$

Explain the identity of these relations from the point of view of the double expansion (ϵ and Fourier).

4. For $P/B^2 \sim \epsilon^2$, $s \sim 1$, show, by substitution, that

$$F = R_0B_0(1 + \varepsilon^2 F_2(r)), \quad F_2(r) = -\varepsilon^2\frac{P}{B_0^2} - \varepsilon^2\int_0^r \frac{dr\ r}{R_0^2}\left(\frac{2-s}{q^2}\right) + O(\varepsilon^4)$$

is a solution of $\frac{1}{2r^2}\left(r^2\psi'^2\right)' + R_0^2P' + FF' = 0$, where we have $\psi' = rF(r)/(R_0q(r))$.

5. What is the toroidal magnetic field and magnetic field strength in a vacuum (no plasma at all)? And compared to the vacuum case, what effect does the plasma pressure and plasma current have on the toroidal field and magnetic field strength near the magnetic axis? (This is sometimes called the plasma response on the toroidal magnetic field.)

6. Using again $\psi' = rF(r)/(R_0 q(r))$, show (e.g. by substitution) that the leading order solution of

$$\Delta'' + \left(2\frac{\psi''}{\psi'} + \frac{1}{r}\right)\Delta' - \frac{1}{R_0} + 2\frac{rR_0 P'}{(\psi')^2} = 0$$

is

$$\Delta'(r) = \frac{r}{R_0}\left[\beta_p(r) + \frac{l_i(r)}{2}\right]$$

where

$$\beta_p(r) = -2\frac{R_0^2 q^2}{B_0^2 r^4}\int_0^r dr\, r^2 P', \quad \text{and} \quad l_i(r) = 2\frac{q^2}{r^4}\int_0^r dr\, \frac{r^3}{q^2}$$

Explain the physical significance of $\Delta(r)$.

7. Show that to leading order, the shaping equation

$$\left[S_m'' + \left(2\frac{\psi''}{\psi'} + \frac{1}{r}\right)S_m' + \frac{1-m^2}{r^2}S_m\right] = 0$$

can be written as

$$r^2 S_m'' + [3 - 2s(r)]rS_m' + (1-m^2)S_m = 0 = 0$$

8. What is the general solution of the shaping equation,

$$r^2 S_m'' + [3 - 2s(r)]rS_m' + (1-m^2)S_m = 0$$

assuming an unsheared q profile (i.e. q'=0).

9. Continuing the last question, obtain the solution with application of boundary conditions $S_m(0) = 0$, and $S_m(a)$ (constants), where a is the edge of the plasma. In particular comment on the radial penetration (from the plasma edge going inwards (reducing r)) of elongation $\kappa(r)$ and triangularity $\delta(r)$, where these are respectively

$$\kappa = \frac{r - S_2}{r + S_2}, \quad \delta = \frac{4S_3}{r},$$

10. On assuming the Jacobian

$$\mathcal{J}_\omega = rR_0\left\{1 + \varepsilon(\epsilon - \Delta')\cos\omega + \varepsilon\sum_{m=2}^{\infty}\left(S_m' - (m-1)\frac{S_m}{r}\right)\cos(m\omega) + O(\varepsilon^2)\right\}$$

and

$$R = R_0\left(1 + \epsilon\cos\omega + O(\varepsilon^2)\right)$$

show that the transformation to straight field line coordinates θ is

$$\theta = \omega - (\epsilon + \Delta')\sin\omega + \sum_m\left[(1-m)\frac{S_m}{r} + S_m'\right]\frac{\sin m\omega}{m} + O(\epsilon^2).$$

Here we have used (from the lecture notes),

$$\frac{d\theta}{d\omega} = \frac{\mathcal{J}_\omega}{\mathcal{J}_\theta}$$

where $\mathcal{J}_\theta = rR^2/R_0$.

11. Optional question: calculate the transformation (to the straight field line angle) to the next order in ϵ, by adopting the Jacobian's at the next order (see lecture notes for definition).

12. Optional question (requires a bit of algebra, and some care): from

$$\mathcal{J}_\omega = R\left(\frac{\partial R}{\partial r}\frac{\partial Z}{\partial \omega} - \frac{\partial R}{\partial \omega}\frac{\partial Z}{\partial r}\right), \quad g_{r,\omega} = \frac{\partial Z}{\partial r}\frac{\partial Z}{\partial \omega} + \frac{\partial R}{\partial r}\frac{\partial R}{\partial \omega}, \quad g_{\omega,\omega} = \left(\frac{\partial Z}{\partial \omega}\right)^2 + \left(\frac{\partial R}{\partial \omega}\right)^2.$$

and

$$R(r,\omega) = R_0\left(1 + \varepsilon\,\epsilon\cos\omega - \varepsilon^2\frac{\Delta(r)}{R_0} + \varepsilon^2\sum_{m=2}^{\infty}\frac{S_m(r)}{R_0}\cos(m-1)\omega + \varepsilon^3\frac{\mathcal{P}(r)}{R_0}\cos\omega + O(\varepsilon^4)\right) \quad (1)$$

$$Z(r,\omega) = R_0\left(\varepsilon\,\epsilon\sin\omega - \varepsilon^2\sum_{m=2}^{\infty}\frac{S_m(r)}{R_0}\sin(m-1)\omega + \varepsilon^3\frac{\mathcal{P}(r)}{R_0}\sin\omega + O(\varepsilon^4)\right), \quad (2)$$

show that

$$\mathcal{J}_\omega = rR_0\left\{1 + \varepsilon(\epsilon - \Delta')\cos\omega + \varepsilon\sum_{m=2}^{\infty}\left(S'_m - (m-1)\frac{S_m}{r}\right)\cos(m\omega) + O(\varepsilon^2)\right\}$$

$$g_{r,\omega} = r\left\{\varepsilon\Delta'\sin\omega - \varepsilon\sum_{m=2}^{\infty}\left(S'_m + (m-1)\frac{S_m}{r}\right)\sin(m\omega) + O(\varepsilon^2)\right\}$$

$$g_{\omega,\omega} = r^2\left\{1 - \varepsilon 2\sum_{m=2}^{\infty}(m-1)\frac{S_m}{r}\cos(m\omega) + O(\varepsilon^2)\right\}$$

13. Further optional question: Following the lecture notes, obtain $\mathcal{P}(r)$ and the metric coefficients to the next order in ϵ.