AQFT Written Exam

The written part of the exam consists of two problems. Provide detailed written solutions and return them by the deadline.

Those marked by $*$ are bonus questions: if you skip them you can still get the top grade by answering properly to the other questions, while a wrong answer to them won't be "held against you"

Problem 1: Supercurrent and Goldstino Theorem

Let's consider a QFT with the following local operators

$$J^{\mu}{}_{\alpha}(x)\,, \qquad \psi_{\alpha}(x) \tag{1}$$

where μ and α are, respectively, a Lorentz 4-vector index and a left-handed spinor index.[1] We are interested in finding the spectral representation of the time-ordered 2-point function $\langle 0|\mathrm{T}J^{\mu}{}_{\alpha}(x)\psi_{\beta}(x)|0\rangle$ and extract some general lesson on the particle spectrum. To this end, answer the following questions.[2]

(a) By inserting a complete set of states in $\langle 0|J^{\mu}{}_{\alpha}(x)\psi_{\beta}(x)|0\rangle$, what are the possible values taken by (j,σ) for the intermediate massive states

$$\langle 0|J^{\mu}{}_{\alpha}(x)|kj\sigma;q\rangle \cdot \langle kj\sigma;q|\psi_{\beta}(y)|0\rangle \neq 0 \quad \Longrightarrow (j,\sigma) = ? \tag{2}$$

to contribute non-vanishingly to the 2-point function?

Likewise, what values are allowed for the helicity λ in the intermediate massless states

$$\langle 0|J^{\mu}{}_{\alpha}(x)|k\lambda;q\rangle \cdot \langle k\lambda;q|\psi_{\beta}(y)|0\rangle \neq 0 \quad \Longrightarrow \lambda = ? \tag{3}$$

that contribute non-trivially to the 2point-function?

[1] The μ-index is an actual 4-vector index, not just up to gauge transformations. In the following, right-handed spinor indices will be represented with the dotted-index convention.

[2] We also adopt the convention where $|kj\sigma;q\rangle$ and $|k\lambda;q\rangle$ represent, respectively, massive and massless Poincaré-irreps of 4-momentum k^{μ}. The massive states carry spin-j and its projection σ, whereas the massless states are labelled by the helicity λ. Possible internal–Poincaré invaraint– quantum numbers are collectively denoted by the "charge" q. The CPT-image of those states carries charge $\bar{q}$. The $|0\rangle$ represents the vacuum state which is, as usual, annihilated by translation and Lorentz generators, and invariant under CPT.

(b) Restricting for now to the massive case and to values of j determined in the previous point, there are two (why?) tensors (with respect to Lorentz and little-group indices) that solve all kinematic constraints demanded by Poincaré symmetry for $\langle 0|J^{\mu}{}_{\alpha}(0)|kj\sigma;q\rangle$ Determine such a tensor structures (up to overall Poincaré-invariant factors $z_{1,2}(m^2,q)$) in terms of Pauli-matrices $\sigma^{\mu}_{\alpha\dot\beta}$, momentum k^{μ}, and the usual wavefunctions[3] $u^{\sigma}_{\alpha}(k)$ and $v^{\dot\alpha\sigma}(k)$.

Hint: you may find convenient to organize the search of the structures in terms of increasing number of insertions of Pauli matrices. Alternatively, you can try to write down tentative local operators in terms of free ψ and/or $\psi^{\dagger}$, their derivatives, and Pauli matrices, so that they would give rise to such non-vanishing matrix elements, whose structure is after all dictated by kinematics, rather than the dynamics that enter only through the $z_i(m^2,q)$.

(c) Restrict now the previously found tensor structures for $\langle 0|J^{\mu}{}_{\alpha}(0)|kj\sigma;q\rangle$ by demanding the additional condition of $J^{\mu}{}_{\alpha}(x)$ being conserved, namely

$$\partial_{\mu}J^{\mu}{}_{\alpha}(x) = 0\,. \tag{4}$$

We referred to such a conserved operator as the *"supercurrent"*.

(d) In order to simplify further the expression for $\langle 0|J^{\mu}{}_{\alpha}(x)\psi_{\beta}(x)|0\rangle$ which can be obtained by inserting intermediate massive states, determine the explicit expression of the the spin sums of the wavefunctions

$$\sum_{\sigma} v^{\sigma\dot\beta}(k)v^{*\sigma\beta}(k) = ?\,, \qquad \sum_{\sigma} u^{\sigma}_{\alpha}(k)v^{*\sigma}_{\beta}(k) = ? \tag{5}$$

(up to an overall phase in the second sum, which is somewhat conventional (related to a phase in the CPT transformation).

Hint: you may perhaps find convenient to determine these spins sum first at a reference momentum $\bar k = (m,0,0,0)$, and then go to a generic momentum with a canonical transformation $L_{\mu}{}^{\nu}\bar k_{\nu} = k_{\mu}$.

(e) Defining the momentum space time-ordered[4] 2-point correlator

$$\langle \mathrm{T}J^{\mu}{}_{\alpha}(k)\psi_{\beta}(-k)\rangle \equiv \int d^4x e^{ikx}\langle \mathrm{T}J^{\mu}{}_{\alpha}(x)\psi_{\beta}(0)\rangle\,, \tag{6}$$

[3]We recall that the wave-functions $u^{\sigma}_{\alpha}(k)$ and $v^{\dot\alpha\sigma}(k)$ are tensors transforming, under Lorentz and little-group, respectively, like the matrix elements $\langle 0|\psi_{\alpha}(0)|kj\sigma;q\rangle$ and $\langle 0|\psi^{\dagger\,\dot\alpha}(0)|kj\sigma;q\rangle$ of a free theory. In particular, they solve the Dirac equation $k_{\mu}\sigma^{\mu}v^{\sigma} = mu^{\sigma}$ and $k_{\mu}\bar\sigma^{\mu}u^{\sigma} = mv^{\sigma}$. The analog wave-functions $u^{\lambda}_{\alpha}(k)$ and $v^{\dot\alpha\lambda}(k)$ of massless states solve instead the Weyl equations.

[4]We recall that time-ordering of fermions $A(x)$ and $B(0)$ is defined with a relative minus sign, that is as $TA(x)B(0) = \theta(x^0)A(x)B(0) - \theta(-x^0)B(0)A(x)$, where $\theta(x)$ is the Heaviside step function.

determine its spectral representation

$$\langle \mathrm{T} J^{\mu}{}_{\alpha}(k)\psi_\beta(-k)\rangle = \int_{M^2}^{\infty} dm^2 \frac{i}{k^2-m^2+i\epsilon}\{\quad ? \quad\}^{\mu}{}_{\alpha\beta} \tag{7}$$

assuming both the mass spectrum is gapped, i.e. $k^2 \equiv m^2 \geq M^2 > 0$, and $J_{\mu\alpha}$ is a conserved supercurrent, Eq. (4). (Note the relative phase between the two spectral densities will be completely fixed by current conservation).

Hint: you may want to use CPT invariance: there is an anti-unitary operator U_{CPT} *such that*

$$U_{\mathrm{CPT}}^{-1}\psi_\alpha(0)U_{\mathrm{CPT}} = \psi^{\dagger}_{\dot\alpha}(0)\,, \quad U_{\mathrm{CPT}}^{-1}J_{\mu\alpha}(0)U_{\mathrm{CPT}} = -J^{\dagger}_{\mu\dot\alpha}(0) \tag{8}$$

$$U_{\mathrm{CPT}}|kj\sigma;q\rangle = |kj-\sigma;\bar q\rangle(-1)^{\sigma+j}\times\eta\,, \quad U_{\mathrm{CPT}}|k\lambda;q\rangle = |k-\lambda;\bar q\rangle(-1)^{\lambda+1/2}\times\eta \tag{9}$$

where and η*'s are* σ*- and* λ*-independent conventional phases.*

(f) * Coming now to intermediate massless states denoted as $|k\lambda;q\rangle$, what are the tensor structures, consistent with Lorentz and little-group transformations, for the matrix elements $\langle 0|J^{\mu}{}_{\alpha}(0)|k\lambda;q\rangle$, compatibly with a non-vanishing product $\langle 0|J^{\mu}{}_{\alpha}(x)|k\lambda;q\rangle \cdot \langle k\lambda;q|\psi_\beta(x)|0\rangle \neq 0$?

Does requiring (super)current conservation (4) enforce further constraints?

(g) * Repeat items (d) and (e) including now intermediate massless states as well in the spectral decomposition. That is, assume now the spectrum is instead gapless, $m=0$ is allowed.

(h) * Finally, let's consider the following interesting limit[5] for the 2-point function of the conserved supercurrent:

$$\lim_{k\to 0} k^{\mu}\langle \mathrm{T} J^{\mu}{}_{\alpha}(k)\psi_\beta(-k)\rangle \equiv \langle 0|\delta_\alpha\psi_\beta|0\rangle\,. \tag{10}$$

Show that whenever the limit is non-vanishing, $\langle 0|\delta_\alpha\psi_\beta|0\rangle \neq 0$, the spectrum of the theory must necessarily contain $|k\lambda = 1/2;q\rangle$, that is a massless helicity $\frac{1}{2}$ particle, with non-vanishing overlap with the supercurrent, i.e. $\langle 0|J_{\mu\alpha}|k\lambda=1/2;q\rangle \neq 0$.

This massless particle goes under the name of "Goldstino", and its existence under the conditions stated is known as the "Goldstino theorem", in analogy with the familiar Goldstone theorem.[6]

[5]It corresponds to an integrated Ward identity $\lim_{k\to 0}\int d^4x e^{ikx}\partial_\mu\langle \mathrm{T} J^{\mu}{}_{\alpha}(x)\psi_\beta(0)\rangle$.

[6]For your own knowledge, the (10) is an integrated Ward identity associated to the so-called supersymmetric transformation generated by exponentiating the supercurrents $Q_\alpha = \int d^3x J^0{}_\alpha$, contracted with a Grassmanian spinor ξ^α which plays the role of Lie parameters. In fact, $\langle 0|\delta_\alpha\psi_\beta|0\rangle = i\langle 0|\{Q_\alpha,\psi_\beta\}|0\rangle$ so that spontaneous supersymmtry breaking, i.e. $Q_\alpha|0\rangle \neq 0 \neq Q^{\dagger}_{\dot\alpha}|0\rangle$ implies the existence of a Goldstone fermion, according to the Goldstino theorem.

Problem 2: Renormalization of Complex Scalar Field Theory

Consider a 4D QFT of two complex scalar fields ϕ_1 and ϕ_2 with Lagrangian (here on, when we write a Lagrangian, we mean the renormalized Lagrangian for some choice of the renormalization scale μ)

$$\mathcal{L} = \left\{ \sum_{i=i}^{i=2} \partial\phi_i^* \partial\phi_i - m_i^2 \phi_i^* \phi_i - \frac{\lambda_i}{4} (\phi_i^* \phi_i)^2 \right\} + \lambda_{12} (\phi_1^* \phi_1)(\phi_2^* \phi_2) + \frac{1}{3!} \left(\lambda \phi_1 \phi_2^3 + \text{h.c.} \right) \tag{11}$$

a) Discuss the global symmetries and renormalization (for instance: are terms not appearing in the eq. (11) necessary for renormalization?). Derive the full set of RG equations at 1-loop.

b) Consider adding to the above lagrangian a small perturbation of the form

$$\Delta\mathcal{L} = \frac{\eta_1}{4!} \phi_1^4 + \frac{\eta_1^*}{4!} (\phi_1^*)^4 \tag{12}$$

with η_4 a complex coupling.

- Discuss the symmetries and, taking them into account, discuss what changes in the renormalization? Which new counterterms, if any, are needed? [7]
- Compute the full set of 1-loop RG equations
- Compute the (in principle matrix) anomalous dimension for the set of fields $(\phi_1, \phi_1^*, \phi_2, \phi_2^*)$ at the lowest *non-trivial* order.

c) Consider adding instead to eq. (11) a small perturbation of the form

$$\Delta\mathcal{L} = \frac{\eta_2}{4!} \phi_2^4 + \frac{\eta_2^*}{4!} (\phi_2^*)^4 \tag{13}$$

with η_4 a complex coupling.

- Discuss the symmetries and, taking them into account, discuss what changes in the renormalization? Which new counterterms, if any, are needed?
- Compute the matrix anomalous dimension for the set of fields $(\phi_1, \phi_1^*, \phi_2, \phi_2^*)$ at the lowest *non-trivial* order.

[7] Here and later you may find it useful to assign to the new couplings transformation properties under the original symmetry in such a way that $\Delta\mathcal{L}$ remains *formally* invariant.

d*) Consider finally the addition of three Dirac fermions ψ_1, ψ_2 and ψ_3, such that the starting Lagrangian is given by the sum of eq. (11) and

$$\mathcal{L}_\psi = \sum_{a=1}^{a=3} \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a + \left[y\phi_2(\bar{\psi}_1\psi_2 + \omega\bar{\psi}_2\psi_3 + \omega^2\bar{\psi}_3\psi_1) + \text{h.c.} \right] \tag{14}$$

with $\omega = e^{i2\pi/3}$ is the cubic root of the identity and y a complex Yukawa coupling.

- Discuss symmetry and renormalization in general. Which counterms are needed?
- Derive the RG equations at 1-loop.