The spherical p-spin solved with replica

- Compute the complexity
- Replica symmetry breaking
  and its physics

Computing the complexity from cloning

Reminder of the clone method:

$$Z_N^{(m)} = \int df e^{N\Sigma(f)} e^{-m\beta f N} \equiv e^{-\beta m \phi(m) N}$$

$$-m\beta\phi(m) = \max_{f|\Sigma(f)\geq 0} [\Sigma(f) - m\beta f]$$

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Parametric determination of complexity from $\phi(m)$:

$$f = \frac{d(m\phi)}{dm}$$

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$$f = \frac{d(m\phi)}{dm}$$

$$\Sigma = \beta m^2 \phi'(m)$$

$$\begin{aligned}\Sigma(f) &= \min_m [-m\beta\phi(m) + m\beta f] \\ &= -\beta m[\phi(m) - f]|_{d(m\phi)/dm=f} \\ &= -\beta m \left[\phi(m) - \frac{d(m\phi)}{dm}\right] |_{d(m\phi)/dm=f} \\ &= \beta m^2 \phi'(m)|_{d(m\phi)/dm=f}\end{aligned}$$

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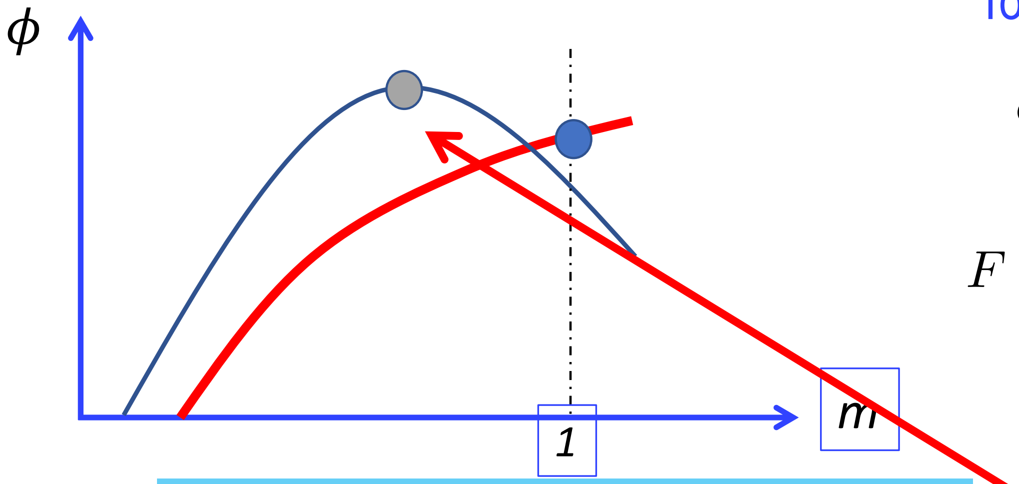

Physical range of ϕ : $\Sigma > 0 \;\leftrightarrow\; \phi' > 0$
and $\Sigma''(f) < 0 \leftrightarrow (m\phi)'' < 0$

Total quenched free energy:

$$e^{-\beta F} = \int_{\Sigma(f)\geq 0} df e^{N(\Sigma(f)-\beta f)}$$

$$F = \min_{f|\Sigma(f)\geq 0} \left[f - \beta^{-1}\Sigma(f)\right] = \phi(1),$$
if $\Sigma(f_{\min}) \sim \phi'(1) \geq 0$,
else
$$= f|_{\Sigma(f)=0} = \phi|_{\phi'=0} = \max_m \phi(m).$$

$$F = F_{\text{qu}} = \max_{m\leq 1} \phi(m)!$$

Cloned free energy of spherical p-spins with replicas

$$Z^{(m)} = \exp(-\beta N \Phi(m)) = ? \qquad \Phi(m) \equiv m\phi(m) = ?$$

$$H = H_J[\sigma_1] + \cdots + H_J[\sigma_m] - \epsilon \sum_{a,b}^{1,m} \sum_{i=1}^{N} \sigma_i^a \sigma_i^b$$

Clone forming attraction
(dropped in the end)

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$$D\sigma \;\breve{=}\; \left(\prod_i d\sigma_i\right) \delta\left(\sum_i \sigma_i^2 = N\right)$$

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For integer n:

$$\overline{(Z_m)^n} = \overline{\int D\sigma_1 \cdots D\sigma_{nm} e^{-\beta(H_J[\sigma_1]+\cdots+H_J[\sigma_{nm}])}}$$

n x m copies!

Cloned free energy of spherical p-spins with replicas

$a = 1, ..., nm$

$$\overline{(Z_m)^n} \propto \int D\sigma_i^a \prod_{i_1 < \cdots < i_p} \int dJ_{i_1 \cdots i_p} \exp\left[-J^2_{i_1 \cdots i_p} \frac{N^{p-1}}{p!} + \beta J_{i_1 \cdots i_p} \sum_{a=1}^{mn} \sigma^a_{i_1} \cdots \sigma^a_{i_p} \right]$$

Product over all p-tuples
(clone attraction is now not explicitly written)

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$$\propto \int D\sigma_i^a \prod_{i_1<\cdots<i_p} \exp\left[\frac{\beta^2 p!}{4N^{p-1}}\sum_{a,b}^{1,mn}\sigma^a_{i_1}\sigma^b_{i_1}\cdots\sigma^a_{i_p}\sigma^b_{i_p}\right]$$

Gaussian average over independent couplings

Get rid of disorder!

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Gaussian average over independent couplings

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$$= \int D\sigma_i^a \exp\left[\frac{\beta^2}{4N^{p-1}}\sum_{a,b}^{1,mn}\left(\sum_i^N \sigma_i^a\sigma_i^b\right)^p\right] = \int D\sigma_i^a \exp\left[N\frac{\beta^2}{4}\sum_{a,b}^{1,mn}\left(\frac{1}{N}\sum_i \sigma_i^a\sigma_i^b\right)^p\right]$$

Crossterms with identical indices are subleading by O(1/N)

Overlap (global similarity) between replica a and b :

$$Q(\sigma^a,\sigma^b) = \frac{1}{N}\sum_i \sigma_i^a\sigma_i^b$$

Cloned free energy of spherical p-spins with replicas

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$$= \int dQ \exp\left[N\frac{\beta^2}{4}\sum_{a,b}^{1,mn}Q^p_{ab}\right]\int d\sigma_i^a \prod_{a\le b}^{1,mn}\delta\left(NQ_{ab}-\sum_i\sigma_i^a\sigma_i^b\right)$$

$dQ = \prod_{a<b} dQ_{ab}$ and $Q_{aa}=1$

Cloned free energy of spherical p-spins with replicas

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$dQ = \prod_{a<b} dQ_{ab}$ and $Q_{aa} = 1$

$$J(Q) = \int d\sigma_i^a \prod_{a\le b}^{1,mn}\delta\left(NQ_{ab}-\sum_i\sigma_i^a\sigma_i^b\right) = \int d\vec{\sigma}^a\,\delta(NQ_{ab}-\vec{\sigma}^a\cdot\vec{\sigma}^b)$$

Jacobian J(Q)

$$J(Q) = \int d\sigma_i^a \prod_{a\leq b}^{1,mn} \delta\left(NQ_{ab} - \sum_i \sigma_i^a \sigma_i^b\right)$$

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Important:

Virtue of all-to-all (mean field) setting:
Different sites have been decoupled!
Only single-site interactions between the mn replica $\sigma_i^{a=1,\ldots,mn}$

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Saddle point wrt $\lambda_{ab} \longrightarrow (\lambda_*^{-1})_{ab} = Q_{ab}$

$$J(Q) = const \cdot \int d\sigma \exp\left(nmN - \sum_{a\leq b} Q_{ab}^{-1} \sum_{i=1}^{N} \sigma_i^a \sigma_i^b\right) = const \cdot [\det Q]^{N/2}$$

Cloned free energy of spherical p-spins with replicas

→
$$\overline{(Z_m)^n} \propto \int dQ_{ab} \; e^{NX(Q)} \; ,$$

$$X(Q) = \frac{\beta^2}{4} \sum_{ab} Q_{ab}^p + \frac{1}{2} \log \det Q$$

Due to mean field structure:
Final integral over global replica overlaps Q_{ab} , with an action ~ N

→ Saddle point over Q_{ab} ! ?

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But recall:

$$\Phi(m,T) = -T \lim_{N\to\infty}\frac{1}{N}\lim_{n\to 0}\partial_n \int dQ_{ab}\ \exp\left[NX(Q)\right]$$

Saddle point requires exchange of limits to $n \to 0, \quad N \to \infty$!

Find a saddle point Q^*_{ab} for any m,n!

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Find a saddle point Q^*_{ab} for any m,n! & Make sure the second derivatives of $X(Q^*)$ are negative!

Cloned free energy of spherical p-spins with replicas

Recall clone coupling in blocks (B) of m spins:

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Saddle point equation for Q_{ab} is complicated: no general solution

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But: Physical guess of a sensible structure (confirmed by exact solution):

- Replicas of the same block are coupled in the same valley-> finite overlap $Q_{a\neq b} = q$, $Q_{aa} = 1$
- a and b in different blocks: uncorrelated $Q_{ab} = 0$

"One-step replica symmetry breaking structure" :

$$Q = \begin{pmatrix} \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix} \end{pmatrix}$$

m=3 (clones)

n=2 (blocks of replica clones) → 0 eventually

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Explicit breaking of replica symmetry (permutation of replica indices)

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$$\det\begin{pmatrix} 1 & q & q \\ q & 1 & q \\ q & q & 1 \end{pmatrix} = (1-q)^{m-1}[1+(m-1)q] \qquad \det Q = \left\{(1-q)^{m-1}[1+(m-1)q]\right\}^n$$

$$X(Q) = -\beta n m \phi_{1\mathrm{RSB}}(m,q,T) + \beta\epsilon n m(m-1)q$$

$$\phi_{1\mathrm{RSB}}(m,q,T) = -\frac{1}{2\beta}\left\{\frac{\beta^2}{2}\left[1+(m-1)q^p\right] + \frac{m-1}{m}\log(1-q) + \frac{1}{m}\log\left[1+(m-1)q\right]\right\}$$

continuation to $n \to 0$ and real m

$$\Phi(m,T) = -T\partial_n X(Q^*) \;= m\phi_{1\mathrm{RSB}}(m,q^*,T) - \epsilon m(m-1)q^*$$

q*: stationary point Choose solution with q* > 0

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continuation to $n \to 0$ and real m

$$\Phi(m,T) = -T\partial_n X(Q^*) = m\phi_{1\mathrm{RSB}}(m,q^*,T) - \epsilon m(m-1)q^*$$

(cf DPRM on tree)

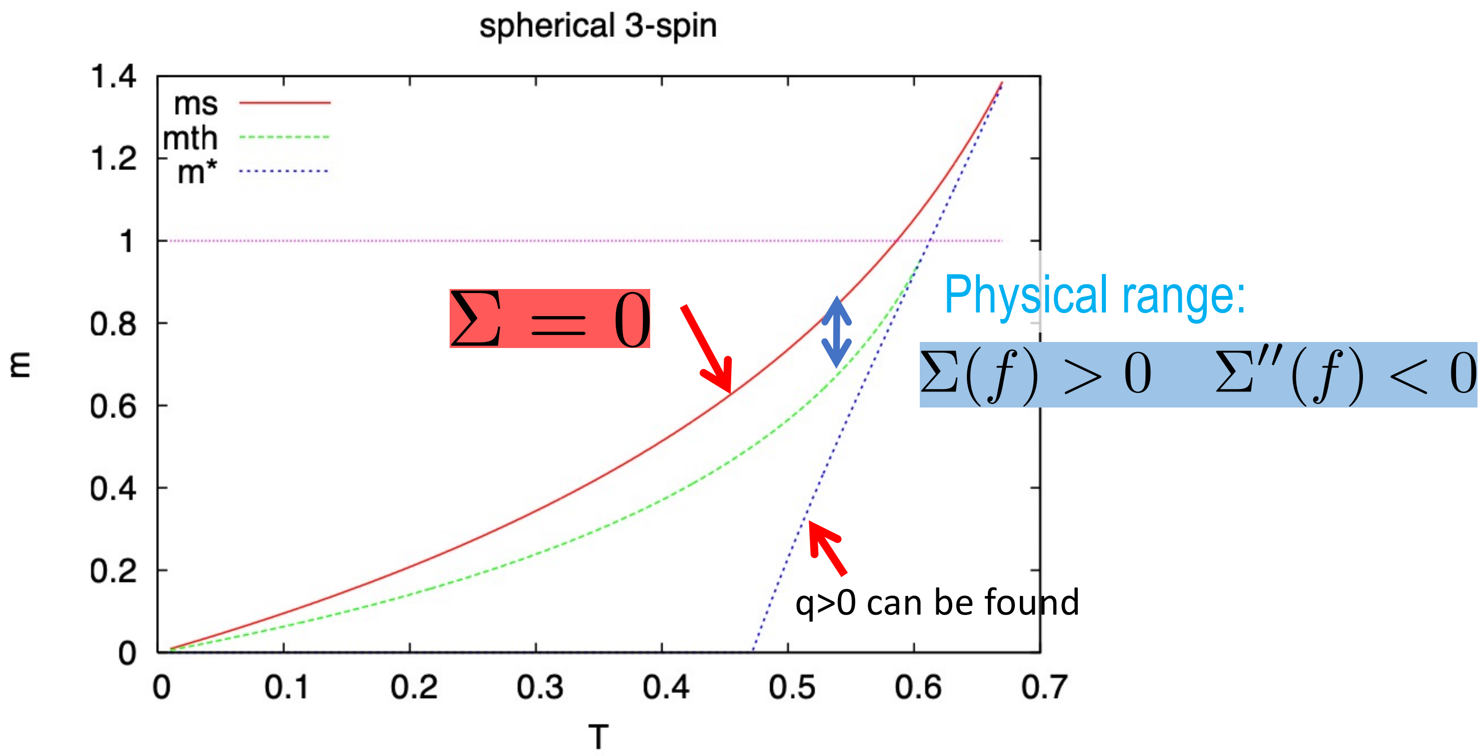
Due to uncorrelated blocks (in this simple model!): Φ = annealed average $\Phi(m,T) = -\frac{T}{N}\log \overline{Z_m}$

Cloned free energy of spherical p-spins with replicas

Having $\Phi(m, T)$

Obtain the spectrum of metastable states!

Cloned free energy of spherical p-spins with replicas



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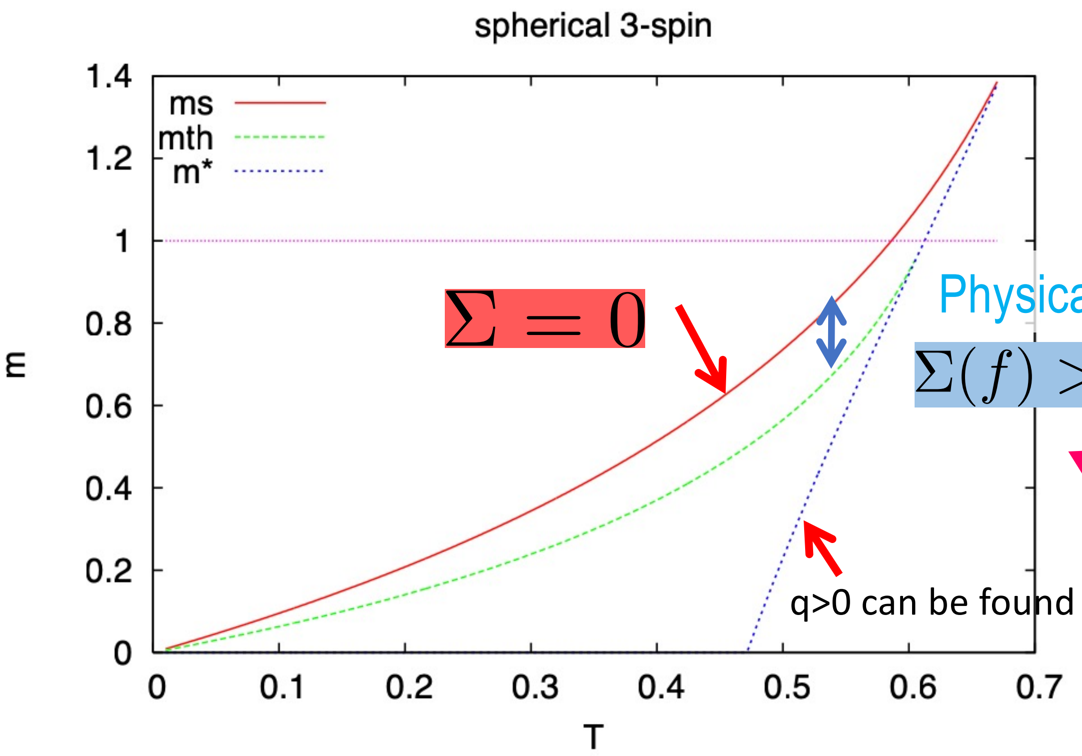
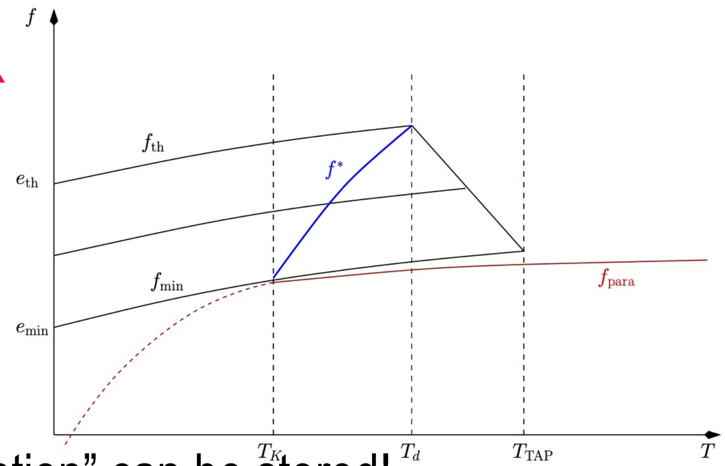


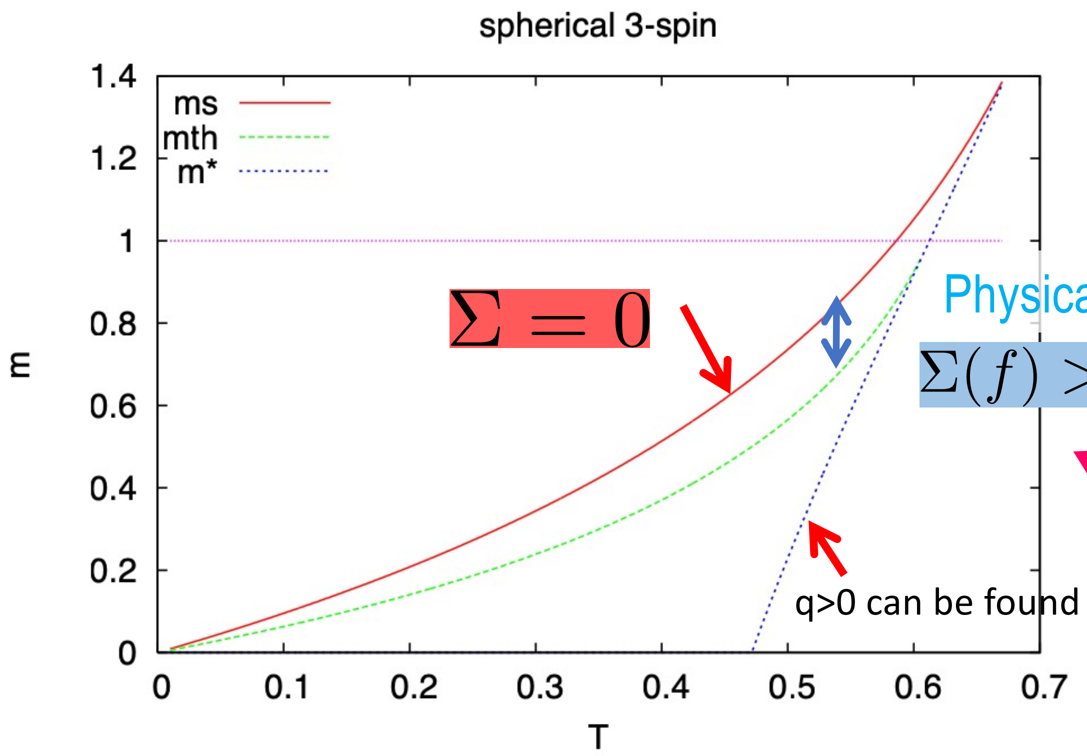
Physical range:

$\Sigma(f) > 0 \quad \Sigma''(f) < 0$



Achieved: full thermodynamics
AND distribution $\Sigma(f)$ of metastable states!
Typical values of $\Sigma \sim 0.01$ (quantify how much "information" can be stored!

Marginal stability of threshold states



$\Sigma = 0$

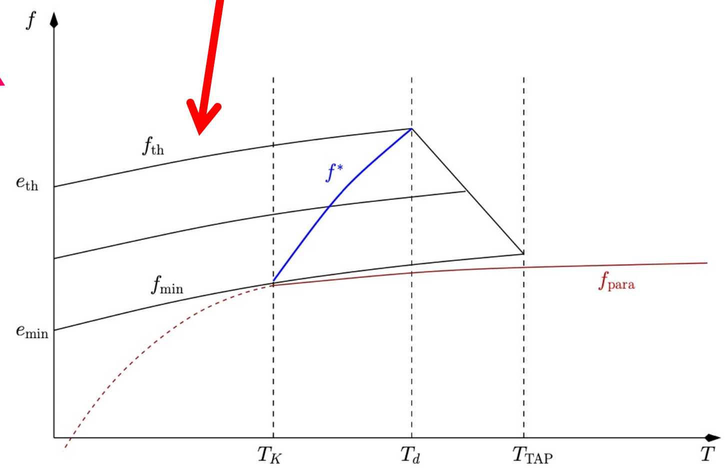
q>0 can be found

Physical range:

$\Sigma(f) > 0 \quad \Sigma''(f) < 0$

Threshold states:

Some eigenvalue of X''(Q), the "replicon" become zero!



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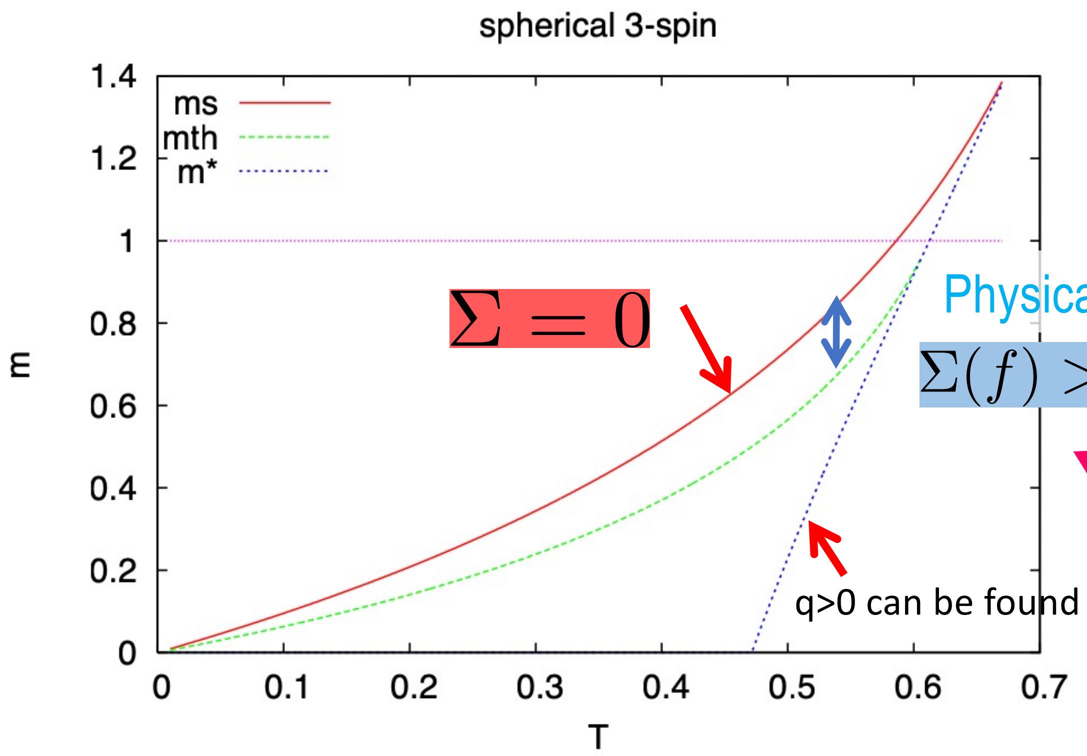


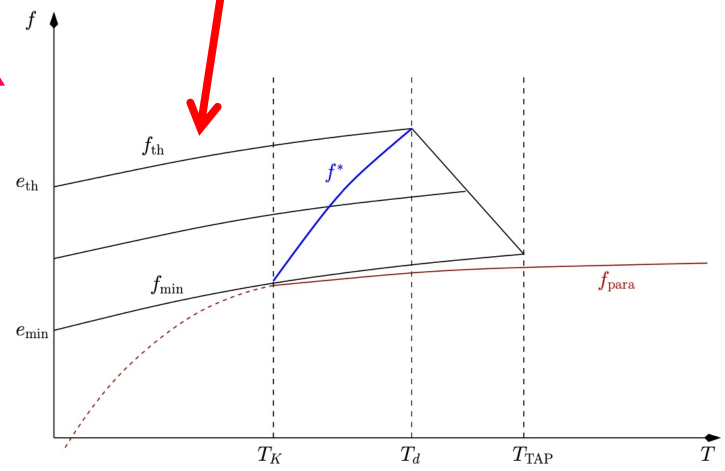
Physical range:

$\Sigma(f) > 0 \quad \Sigma''(f) < 0$

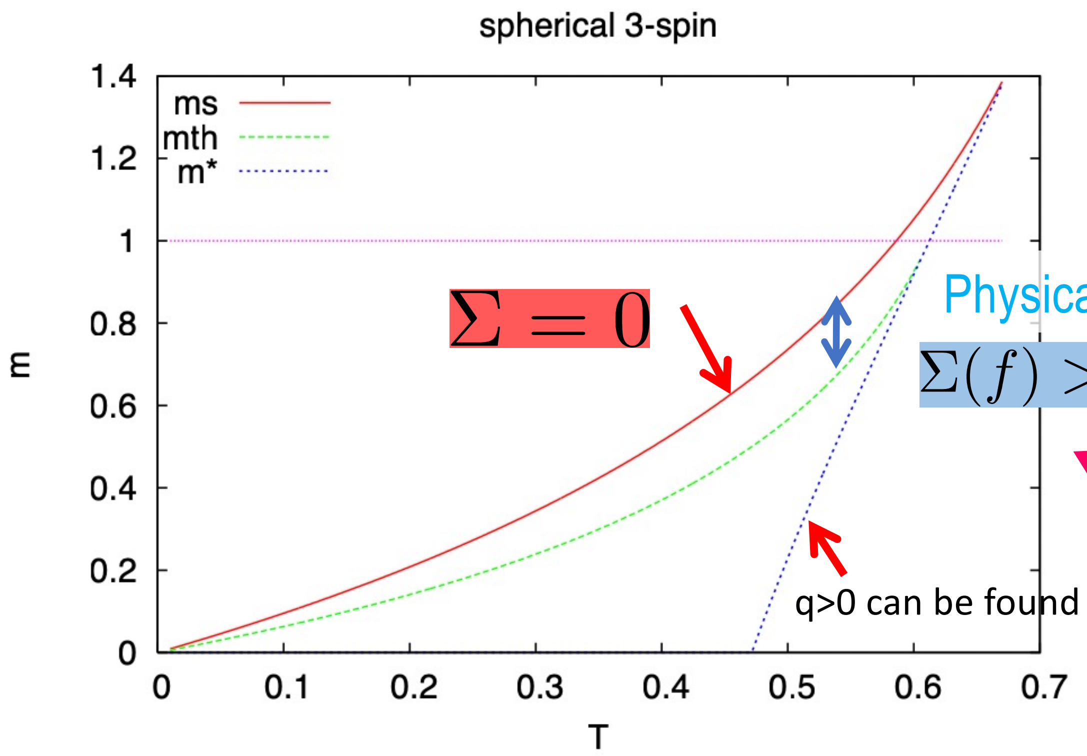
Threshold states:

Marginally stable! TAP minima with gapless fluctuation spectrum

$\partial^2 G(m^{(\alpha)})/\partial m_i \partial m_j$



Marginal stability of threshold states

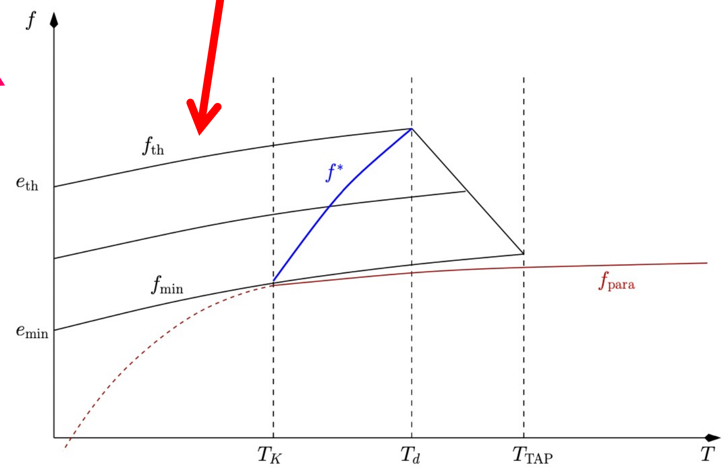


Physical range:

$\Sigma(f) > 0 \quad \Sigma''(f) < 0$

Threshold states:

Marginally stable! TAP minima with gapless fluctuation spectrum $\partial^2 G(m^{(\alpha)})/\partial m_i \partial m_j$



Dynamics (after a quench from high T) floats along the nearly unstable directions, without really settling into the minima

Spontaneous replica symmetry breaking

Clone attraction explicitly breaks permutation symmetry among nm replica

But what about computing for single copy (with no cloning) directly?

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Parisi's proposal (pre-clone!): regard m and q as variational and find stationary point!

$$f_{eq}(T) = \max_{q,0\leq m\leq 1}\phi_{1RSB}(m,q)$$

Imposes the 'equilibrium' choice m* = min(1,m_s) !

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$T < T_K$: $m^* < 1$!
Saddle point chooses a multi-block structure: *Spontaneous* replica symmetry breaking! **"RSB"**

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Meaning of replica symmetry breaking

Meaning of the spontaneous block structure

The different replica lie in the lowest available minima of G(m)

A pair of replica thus may lie in the same minimum (and have overlap q) or in different valleys (no overlap):

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$$\begin{aligned} P(Q_{12}) &\equiv \overline{\delta(Q_{12} - \frac{1}{N}\sum_i \sigma_i^1 \sigma_i^2)} = \lim_{\nu\to 0} \frac{1}{\nu - 1} \sum_{b\neq 1} \delta(Q_{12} - Q_{1b}) \\ &= \lim_{\nu\to 0} \frac{(m-1)\delta(Q_{12}-q) + (\nu - m)\delta(Q_{12} - 0)}{\nu - 1} \\ &= (1-m)\delta(Q_{12}-q) + m\delta(Q_{12}) \end{aligned}$$

Only non trivial for freezing: $m=m_{eq} < 1$
where O(1) **different minima** compete!