Level statistics of random matrices (RMT part II)

Goals: 1) Understand distribution of levels:

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda-\lambda_i) \xrightarrow[N\to\infty]{} \frac{1}{2\pi}\sqrt{4-\lambda^2}$$

"Wigner semicircle" : 3 ways: → cavity approach (indep. matrix elements)
→ classical Coulomb plasma (invariant ensembles)
→ orthogonal polynomials

2) Distribution of level spacings $\delta_n := E_n - E_{n-1}$

so far: $p(\delta) \propto \delta^{\beta}$ $\beta = 1, 2, 4$, for $\delta \ll \Delta$

⇒ full distribution?

3) Two point correlation: prob. to find E_i at distance δ from E_j (possibly with other levels in between)

⇒ interesting correlation hole, like for fermions!

⟶ close analogy with confined fermions

I. Density of levels $\rho(\lambda) := \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i)$

Finite N: ρ fluctuates, but: as $N\to\infty$ $\rho_N(\lambda) \longrightarrow \rho_\infty(\lambda)$ almost surely!

a) Cavity approach for matrices with independent matrix elements:

$\langle H_{ij} H_{kl}\rangle = 0$ unless $(ij) = (kl)$ or (lk)!

levels of H ⇔ eigenvalues of a non-interacting hopping Hamiltonian

$$\mathcal{H} = -\sum_{ij} c_i^{+} H_{ij} c_j$$

To obtain the level distribution ≡ density of states: consider the local Green's function (⇔ Anderson localization!)

$$G_{ii}(E+i\varepsilon) = \langle i | \frac{1}{E+i\varepsilon-\mathcal{H}} | i\rangle \quad ; \text{ for } \mathcal{H}=0: \; G_{ii,0}^{(0)} = \langle i | \frac{1}{E+i\varepsilon} | i\rangle = 1/(E+i\varepsilon)$$

Insert set of eigenstates: $\mathbb{1} = |\lambda\rangle\langle\lambda|$

$$G_{ii}(E+i\epsilon) = \sum_\lambda \langle i|\lambda\rangle \frac{1}{E-E_\lambda+i\epsilon} \langle\lambda|i\rangle$$

$$\Rightarrow \operatorname*{Im}_{\epsilon\to 0} G_{ii}(E+i\epsilon) = \sum_\lambda |\langle i|\lambda\rangle|^2 \left(-\cancel{i}\pi\, \delta(E-E_\lambda)\right)$$

$$\Rightarrow \boxed{\sum_i \operatorname{Im} G_{ii}(E+i\epsilon) \xrightarrow[\epsilon\downarrow 0]{} -\pi N \rho(E)} \quad \text{where } \rho(E) = \frac{1}{N}\sum_\lambda \delta(E-E_\lambda)$$

level density

As $N\to\infty$ expect that $G_{ii} \longrightarrow \overline{G_{ii}} \equiv G$ = average local Green's function

(all sites are equivalent in the type of environment they see)
No onsite disorder!

Recall recursion on Cayley tree:

$$G_{ii}(E) = \frac{1}{G^{-1}_{ii,0}(E) - \Sigma_{ii}(E)} = \frac{1}{E+i\epsilon - \sum_{j(\neq i)} |H_{ij}|^2 G^{(i)}_{jj}(E+i\epsilon)}$$

Gaussianity not needed

~~As~~ $N\to\infty$: $\Sigma_{ii} \longrightarrow \sum_{j(\neq i)} \overline{H_{ij}^2} \cdot \overline{G^{(i)}_{jj}} \simeq N\cdot \overline{H_{ij}^2} \cdot G \equiv h^2 G$

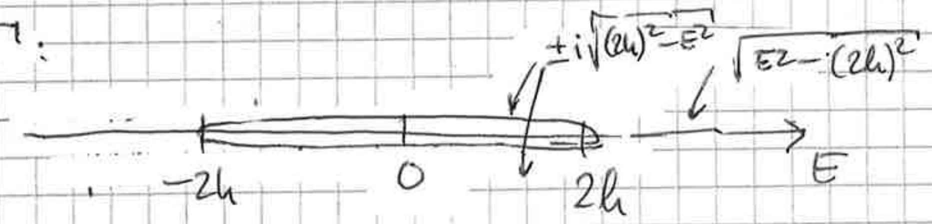
irrelevant for $N\to\infty$

$$\Rightarrow \boxed{G = \frac{1}{E+i\epsilon - h^2 G}} \Leftrightarrow (E+i\epsilon)G - h^2 G^2 = 1$$

$$\Rightarrow G = \frac{E+i\epsilon - \sqrt{(E+i\epsilon)^2 - 4h^2}}{2h^2}$$

ensures that $G(E\to\infty) \sim \frac{1}{E}$!

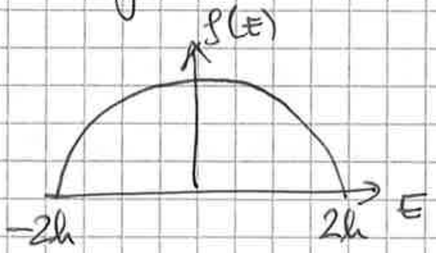
$\sqrt{}$:



$$\Rightarrow \rho(E) = -\frac{1}{\pi}\operatorname{Im}(G(E+i\epsilon)) = \begin{cases} 0 & |E| > 2h \\ \frac{1}{2\pi h^2}\sqrt{(2h)^2 - E^2} & |E| < 2h \end{cases}$$

usual convention: $h^2 \equiv N\overline{H_{ij}^2} \equiv 1$

$\Rightarrow$ Wigner semicircle density of states: $(h^2=1)$



$$\rho(E) = \frac{1}{2\pi}\sqrt{4-E^2}$$

normalization: $\int dE\, \rho(E) = 1$ ✓

Remarks: • $\rho(E)$ is self-averaging! (limit $N\to\infty$ concentrates on the average)

• The semicircle appears universally, as long as off-diagonals are independently distributed with finite variance, $\langle H_{ij}^2\rangle = \frac{h^2}{N}$.

• Interesting to know: for complex, non-Hermitian matrices, one gets complex eigenvalues. They concentrate on a circle around the origin with uniform density.

($\Leftrightarrow$ Re λ and Im λ are distributed with a semicircle D.O.S.)

• The semicircle is not superuniversal: non-Gaussian invariant ensembles differ from the above $\rho(\lambda)$, as we see below.

b) Coulomb plasma approach

Recall: The joint p.d.f. of all energies takes the form:

standard choice for energy unit!

$$P(\{E_n\}) = \text{cst.} \prod_{n>m} |(E_n - E_m)|^\beta \exp\Big(-\sum_n \underbrace{\tfrac{\beta}{2} E_n^2}_{V(E_n)}\Big)$$

$$\equiv \text{cst.}\exp\left[-\beta \mathcal{L}\right]$$

where $\mathcal{L} = -\sum_{n>m} \ln|E_n - E_m| + \sum_n \frac{1}{\beta}\overbrace{V(E_n)}^{E_n^2/2 \text{ for Gaussian ensemble}}$

$$= -\frac{1}{2}\int dE\, \rho(E)\int dE'\, \rho(E')\ln|E-E'| + \frac{1}{\beta}\int dE\, \rho(E)\, V(E)$$

$$\boxed{\rho(E) = \sum_n \delta(E-E_n)}$$

$\Rightarrow \mathcal{L} = \mathcal{L}[\rho]$

Observe: logarithmic interactions are very long range $\Rightarrow$ mean field situation

$\Rightarrow$ • Replace $\rho(E)$ by ensemble (disorder) average $\rho(E) = \langle \rho(E) \rangle$
• Neglect "thermal fluctuations" of $\mathcal{L}$, and instead minimize $\mathcal{L}[\rho]$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta \rho(E)} = 0 \quad \text{(everywhere where } \rho \neq 0 \text{ !)}$$

Upon taking $\frac{d}{dE}\left(\frac{\delta \mathcal{L}}{\delta \rho(E)}\right)$: $\Rightarrow$ $$\underbrace{\int_{-\infty}^{\infty} \rho(E') \frac{dE'}{E-E'}}_{\text{"Coulomb" repulsion by other particles}} = \underbrace{\frac{1}{\beta}\frac{dV}{dE}}_{\text{confinement force}} \equiv \underbrace{f(E)}_{= E \text{ for standard choice in Gaussian ensembles}} \quad (*)$$

$\Leftrightarrow$ force balance equation on particles at E:

Elegant solution to the Plasma equilibrium equation $(*)$:

[Note: $\frac{1}{E-E'}$ is to be understood as $\frac{1}{2}\left(\frac{1}{E-E'+i0} + \frac{1}{E-E'-i0}\right) \Leftrightarrow$ principal part!]

Split $\rho(E)$ into two parts: $\rho(E) = \rho_+(E) + \rho_-(E)$

analytic in the half plane $\pm \mathrm{Im}(z) > 0$

$\Rightarrow$ LHS of $(*)$ $\Rightarrow$

$$\int_{-\infty}^{\infty} \rho(E') \frac{dE'}{E-E'} = \int \rho_+(E') \left(\frac{1}{2}\frac{1}{E-E'+i0} + \underbrace{\frac{1}{2}\frac{1}{E-E'-i0}}_{0}\right) + \int \rho_-(E')\left(\frac{1}{E-E'}\right)$$

$$= -\pi i \left(\rho_+(E) - \rho_-(E)\right)$$

$(*)$ requires this $\stackrel{!}{=} f(E)$ wherever $\rho(E) \neq 0$.

Expect: $\rho(E) \neq 0$ for $|E| < D$, but $= 0$ for $|E| > D$ for some bandwidth D.

$\Rightarrow$ Ingenious ansatz:

$$\rho_\pm(E) = \sqrt{D^2 - E^2} \cdot \int_{-D}^{D} \frac{\varphi(E')\, dE'}{E' - E \mp i0}$$

for $|E| > D \Rightarrow \rho_\pm \propto \pm i \sqrt{E^2 - D^2}$ $\Rightarrow \rho = 0$

analyticity in $\pm \mathrm{Im}(z) > 0$

What should φ be?

Consider $|E| < D$:

$$\text{Evaluate: } \rho_+(E) - \rho_-(E) \overset{\triangle}{=} \frac{i}{\pi} \cdot f(E) = \sqrt{D^2-E^2} \int_{-D}^{D} \varphi(E')\, dE' \underbrace{\left(\frac{1}{E'-E-i0} - \frac{1}{E'-E+i0} \right)}_{= 2\pi i\, \delta(E'-E)}$$

$$= 2\pi i \sqrt{D^2-E^2}\; \varphi(E)$$

$$\Rightarrow \varphi(E) = \frac{f(E)}{2\pi^2 \sqrt{D^2-E^2}}$$

$$\Rightarrow \underline{\text{Solution}}: \quad \rho(E) = \rho_+(E) + \rho_-(E) = \sqrt{D^2-E^2} \cdot \underbrace{\int_{-D}^{D} dE' \, \frac{f(E')}{\pi^2 \sqrt{D^2-E'^2}} \, \frac{1}{E'-E}}$$

onset like Wigner semicircle

in general non-trivial function of E!

Check: For Gaussian ensembles ($\Leftrightarrow$ independent matrix elements)

$$\rightarrow f(E) = E$$

$$\Rightarrow \int_{-D}^{D} \frac{dE'}{\sqrt{D^2-E'^2}} \, \frac{E' + (-E+E)}{E'-E} = \int_{-D}^{D} \frac{dE'}{\sqrt{D^2-E'^2}} + E \int_{-D}^{D} \frac{dE'}{\sqrt{D^2-E'^2}} \, \frac{1}{2}\left(\frac{1}{E'-E+i0} + \frac{1}{E'-E-i0} \right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$= \pi \qquad\qquad 0$$

$$\Rightarrow \rho(E) = \frac{\sqrt{D^2-E^2}}{\pi} \qquad \text{[Gaussian ensemble]}$$

$D = ?$ Here: $\int dE\, \rho(E) \overset{!}{=} N$

$$= \frac{D^2}{\pi} \cdot \frac{\pi}{2} \rightarrow D^2 = 2N \Rightarrow \rho(E) = \frac{\sqrt{2N-E^2}}{\pi}$$

Wigner semi-circle!

Conclusion: Invariant ensembles start as $\rho(E) \underset{E \to D}{\sim} \sqrt{D-E}$

<u>but</u> $\rho(E)$ is <u>not</u> universal; <u>except</u> for Gaussian ensemble (or other random matrices with independent matrix elements

2. Level spacing distribution

$\delta_n := E_n - E_{n-1} \Leftarrow$ subsequent levels $\quad \overline{p(\delta_n)} = ?$

$s := \delta_n/\Delta \quad \Delta = \Delta_E$ mean level spacing $= 1/\rho(E) \Leftarrow$ varies only very slowly with E! on scales $dE \gg \Delta$

a) **Wigner's surmise**

orthogonal ensemble (real H) $\beta = 1$.

Assume: probability of a level to occur at a distance $\in [s', s'+ds']$ from a given level is $\underset{\text{constant}}{a} \cdot \underbrace{s'}_{(s')^{\beta=1}!} ds'$

Assume this holds even for large $s' \gtrsim \Delta$, and different intervals are uncorrelated (this is both wrong!)

$$\Rightarrow \underset{\text{prob. that first level appears in } [s, s+ds)}{P(s)\,ds} = as\,ds \left(\lim_{m\to\infty} \prod_{r=0}^{m-1} \left(1 - \overbrace{\frac{s}{m}}^{ds} \cdot a \cdot s \cdot \frac{r}{m}\right)\right) = as\,ds \exp\left(-\sum_{r=0}^{m-1} \frac{as^2 r}{m^2}\right)$$

$$\xrightarrow[m\to\infty]{} as\,ds\, e^{-as^2/2}$$

Measure level distance in mean level spacing Δ, $\quad s \equiv \frac{(E_n - E_{n-1})}{\Delta}$

$\Rightarrow$ Normalization $\int P(s)\,ds = 1$ ✓

Expected s: $\int s P(s)\,ds = 1 \Rightarrow a = \frac{\pi}{2} \quad (\beta = 1)$

$$P_{\text{Wigner}}(s)\,ds = \frac{\pi}{2} s\, e^{-\pi s^2/4}\, ds$$

(Wigner's surmise) not exact!

Exact for Gaussian 2×2 matrices!

Similar for $\beta = 2, 4$:

$$\text{Approximation: } P_W^{(\beta)}(s) = A_\beta\, s^\beta\, e^{-B_\beta s^2}$$

s^β: known from $P(\{e_n\})$, level repulsion

$e^{-B_\beta s^2}$: see next page!

(pretty convenient interpolation formula!)

Normalization & $\langle s \rangle = 1 \Rightarrow A_\beta = 2 B_\beta^{1+\beta/2}\, \Gamma\left(\frac{\beta}{2}+1\right)$

$B(\beta=1) = \pi/4 \qquad B(\beta=2) = \frac{4}{\pi} \qquad B(\beta=4) = \frac{64}{9\pi}$

b) Exact asymptotics for $P(s \gg \Delta)$ from Coulomb plasma

Probability to find a large gap with no levels inside; around $E=0$ (middle of the spectrum) : $E_n \notin [-\frac{\omega}{2}, \frac{\omega}{2}]$ $\forall n$! $(\omega \gg \Delta)$
$\Rightarrow s \gg 1$

$$\Leftrightarrow \rho(E) = 0 \quad \text{for } |E| < \frac{\omega}{2}$$

$\Rightarrow$ Compute thermodynamic probability : ($\Rightarrow$ "large deviation")

minimize, subject to this additional constraint! $\rho(|E|<\omega) = 0$

$$P_\omega \propto \exp(-\beta\,\delta\mathcal{L}_\omega)$$

$\delta\mathcal{L}_\omega = ?$ Same strategy as before!

Minimizing $\rho \equiv \rho_\omega(E)$:

- Make sure $\rho_\omega(E) = 0$ for $|E| > D$ and $|E| < \omega/2$
- for $\omega \to 0$ reduce to previous $\rho(E)$
- for $\omega > 0$; DOS must increase for $E \gtrsim \frac{\omega}{2}$

$$\Rightarrow \text{Ansatz} : \rho_\omega = \frac{\sqrt{D^2-E^2}\cdot E}{\sqrt{E^2-(\omega/2)^2}} \cdot \left(\int_{-D}^{-\omega/2} + \int_{\omega/2}^{D} dE'\right)\left[\frac{\varphi(E')}{E'-E}\right]$$

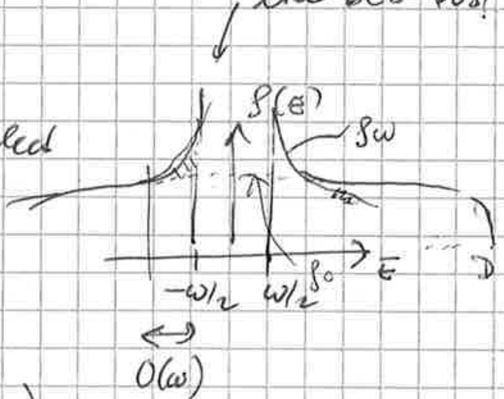
$$\text{now with } \varphi(E') = \frac{f(E')}{\sqrt{D^2-E'^2}} \frac{\sqrt{E'^2-(\omega/2)^2}}{E'}$$

Now, relevant ω are $\propto \Delta \ll D \sim N^{1/2}$

$\Rightarrow$ slow variation of integrand with $E \to$ neglect

$$\Rightarrow \rho_\omega(E) \simeq \rho_0(E) \cdot \frac{|E|}{\sqrt{E^2-(\omega/2)^2}}$$

like BCS-DOS!



$\hookrightarrow$ Insert $\rho_\omega(E)$ in $\delta\mathcal{L}_\omega = \mathcal{L}(\rho_\omega) - \mathcal{L}(\rho_0)$

$\Longrightarrow$ $\delta\mathcal{L}_\omega$ is second order in $\rho_\omega - \rho_0$ $\left(\frac{\delta\mathcal{L}}{\delta\rho} = 0 !\right)$

$\Longrightarrow$ Expect $\delta\mathcal{L}_\omega \sim \omega^2$ ($\log(\omega)$ does not occur)

Computation gives $\delta\mathcal{L}_\omega = \frac{\pi^2}{16}(\rho_0\,\omega)^2 = \frac{\pi^2}{16}\left(\frac{\omega}{\Delta}\right)^2$

$$\Longrightarrow \Longrightarrow P(s) \sim e^{-\beta\cdot\frac{\pi^2}{16}s^2} \quad \Longrightarrow B_\beta^{\text{exact}} = \beta\cdot\pi^2/16 !$$

Exact asymptotics is quite close to Wigner surmise!

Especially for unitary case: $\beta = 2$ $B^{(\beta=2)}_{Wigner} = \frac{4}{\pi} = 1.27$

$\Leftrightarrow$ $B_{exact} = \frac{\pi^2 \beta}{16} = 1.24$

Full calculation of $P(s)$ is possible, but very tedious!

3. Two point function, level compressibility and free fermion analogy

Here we discuss mostly the unitary ensemble and discover a very beautiful connection with free fermions in a parabolic potential.

Observation: $\beta = 2$:

$$P(\{E_n\}) = \prod_{n>m} (E_n - E_m)^2 \, e^{-\frac{\beta}{2}\frac{1}{4}\sum_n E_n^2}$$

$$\stackrel{?}{=} |\psi(E_1, \ldots, E_n)|^2 \quad ? \quad \leftarrow \text{Quantum wave function?}$$

Yes, indeed! $\psi = \overbrace{\prod_{n>m} (E_n - E_m)}^{\equiv \Delta} e^{-\frac{1}{2}\sum_n E_n^2}$

Note: ψ changes sign upon permuting $E_n \Leftrightarrow E_m$

Indeed $\Delta = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ E_1 & E_2 & & E_N \\ E_1^2 & E_2^2 & & E_N^2 \\ E_1^{N-1} & E_2^{N-1} & & E_N^{N-1} \end{vmatrix}$ is a determinant (Vandermond) and thus antisymmetric

$\psi \to$ fermionic wavefunction of N particles with position E_i.

Is ψ the ground state of a known system? Yes!

First: Easy to show: $\sum_i \frac{\partial^n \Delta}{\partial E_i^n} = 0$ $n = 1, 2, \ldots$ (expand the determinant!)

$n = 2 \Rightarrow$ Δ satisfies $\left(-\sum_i \frac{1}{2} \partial^2_{E_i}\right)\Delta = 0$

$\Rightarrow$ zero kinetic energy state of free fermions?

But: Δ is not normalizable $\Rightarrow$ need confining factor $e^{-\sum E_n^2}$

Observe : Δ is invariant upon adding rows with lower index to rows with higher index $\Rightarrow$ make arbitrary polynomials of increasing order !

$$\Delta = \begin{vmatrix} 1 & \cdots & 1 \\ p_1(E_1) & & p_1(E_N) \\ p_2(E_1) & & p_2(E_N) \\ \\ p_{N-1}(E_1) & & p_{N-1}(E_N) \end{vmatrix}$$

where $p_n(E)$ is n'th order polynomial (starting with $1\cdot E^n$)

Well adapted choice : Hermite polynomials !

$$\Rightarrow \boxed{\psi = \Delta \cdot e^{-\sum_n E_n^2/2} = \begin{vmatrix} \varphi_0(E_1) & \cdots & \varphi_0(E_N) \\ \varphi_1(E_1) & \cdots & \varphi_1(E_N) \\ \vdots & & \\ \varphi_{N-1}(E_1) & \cdots & \varphi_{N-1}(E_N) \end{vmatrix}}$$

$\leftarrow$ Slater determinant!

with $\varphi_n(E) = H_n(E)\, e^{-E^2/2}$!

Where $H_n(E)$ are the Hermite polynomials :

$\Rightarrow$ The $\varphi_{n=0,\dots,N-1}$ are the N lowest lying eigenstates of the Hamiltonian

$$H = -\frac{1}{2}\partial_E^2 + \underbrace{\frac{1}{2}E^2}_{\equiv \frac{1}{2}V(E)\,!} \qquad H\varphi_n = \mathcal{E}_n\varphi_n \qquad \mathcal{E}_n = n + \frac{1}{2}$$

(harmonic oscillator)

$\Rightarrow$ ψ is the G. State of N (spinless) fermions in a harmonic potential !

Important property :

The $H_n(E)$ are orthogonal polynomials since

$$\boxed{\int dE\, \varphi_n(E)\varphi_m(E) = \int dE\, e^{-V(E)}\, H_n(E) H_m(E) \equiv \langle H_n, H_m\rangle = h_n \delta_{nm}}$$

NB: For $\beta = 1$ and $\beta = 4$ one may also write

$$\psi_{0,\beta} := \prod_{n>m} |E_n - E_m|^{\beta/2}\, \mathrm{sgn}(E_n - E_m)\, e^{-\sum_n E_n^2/2} \Rightarrow P(\{E\}) = |\psi_{0,\beta}|^2$$

But now $\psi_{0\beta}$ is the G.S. of an interacting problem :

$$H = -\frac{1}{2}\sum_n \partial_{E_n}^2 + \frac{\beta}{2}\left(\frac{\beta}{2}-1\right)\sum_{n>m}\frac{1}{(E_n-E_m)^2}$$

Calogero - Sutherland

$\beta = 2$: $P(\{E_n\}) = |\Psi_{GS}|^2$! $\Rightarrow$ 1 point function $\rho(\lambda)$ $\Leftrightarrow$ fermion density

2 point function $\Leftrightarrow$ density-density correlator!

$\Longrightarrow$ Very powerful!

3.1. Orthogonal polynomials

Systematic construction: Wanted: $\langle H_n, H_m \rangle = h_n \delta_{nm}$. $\Rightarrow$ φ_n are orthogonal wavefunctions $\Rightarrow \Psi$ is a Slater determinant of orthogonal φ_n's!

Start: $H_0(x) = 1 \qquad H_1(x) = x$

✱ Recursion $\boxed{H_{n+1}(x) = x\, H_n(x) - C_n H_{n-1}(x)}$

Note: $H_n(x)$ are even/odd in x, if n is even/odd. $\Rightarrow \langle H_{n+1}, H_n \rangle = 0$

Proof: (induction) $\langle H_{n+1}, H_m \rangle = \langle x H_n - C_n H_{n-1}, H_m \rangle = \langle H_n x H_m \rangle - C_n \langle H_{n-1}, H_m \rangle$

$= 0$ for $m \leq n-2$ by assumption

Remaining condition:

$$\langle H_{n+1}, H_{n-1} \rangle \overset{!}{=} 0 \Leftrightarrow \underbrace{\langle x H_n, H_{n-1} \rangle} = C_n \overbrace{\langle H_{n-1}, H_{n-1} \rangle}^{h_{n-1}}$$

$$= \langle H_n, x H_{n-1} \rangle = \langle H_n, H_n \rangle = h_n$$

✱✱ $\Rightarrow \boxed{C_n = h_n / h_{n-1}}$ ∎.

Orthonormal ~~polynomials~~ wave functions: $\varphi_n(x) = \frac{1}{\sqrt{h_n}} H_n(x)\, e^{-x^2/2}$

NB: This construction is general and does not rely on the existence of the 2nd order Schrödinger equation!

Efficiency of orthogonal polynomials is due to an important relation for the "fermion Green's function": (Formula by Christoffel + Darboux)

$$K_N(x,y) := \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y) = \sqrt{\frac{h_N}{h_{N-1}}}\, \frac{\varphi_{N-1}(x)\varphi_N(y) - \varphi_{N-1}(y)\varphi_N(x)}{y-x}$$

Proof: By induction, using the recursion ✱ and the relation ✱✱.

$$\sqrt{\frac{h_N}{h_{N-1}}}\left(\varphi_{N-1}(x)\varphi_N(y)-\varphi_{N-1}(y)\varphi_N(x)\right)=\frac{e^{-x^2/2-y^2/2}}{h_{N-1}}\left(H_{N-1}(x)H_N(y)-H_{N-1}(y)H_N(x)\right)$$

$$=\frac{e^{-\frac{x^2+y^2}{2}}}{h_{N-1}}\left[H_{N-1}(x)\left(yH_{N-1}(y)-C_{N-1}H_{N-2}(y)\right)-H_{N-1}(y)\left(xH_{N-1}(x)-C_{N-1}H_{N-2}(x)\right)\right]$$

$$=\frac{e^{-\frac{x^2+y^2}{2}}}{h_{N-1}}\left[(y-x)H_{N-1}(x)H_{N-1}(y)+C_{N-1}\left(H_{N-1}(y)H_{N-2}(x)-H_{N-1}(x)H_{N-2}(y)\right)\right]$$

$$=(y-x)\varphi_{N-1}(x)\varphi_{N-1}(y)+\underbrace{\frac{1}{h_{N-2}}\sqrt{h_{N-1}h_{N-2}}\left(\varphi_{N-1}(y)\varphi_{N-2}(x)-\varphi_{N-1}(x)\varphi_{N-2}(y)\right)}_{=(y-x)\sum_{n=0}^{N-2}\varphi_n(x)\varphi_n(y)\text{ by assumption.}}$$

$$=(y-x)\sum_{n=0}^{N-1}\varphi_n(x)\varphi_n(y) \qquad \boxtimes.$$

⇒ The Green's function can be expressed as a simple expression involving only eigenfunctions φ_N with large N!

⇒ Task: Find asymptotic expression for $\varphi_N(x)$!

3.2. Use of the Green's function $K_N(x,y)$:

Fermionic field operator: $\hat{\Psi}(x)=\sum_{n\geq 0}\varphi_n(x)\,c_n$ — annihilates fermion in nth eigenstate φ_n

Orthogonality ⇒ $\rightarrow c_n^+c_m+c_mc_n^+=\delta_{nm}$ fermionic anti-commutation

$$|0\rangle \equiv |\Psi_{GS}\rangle=\prod_{n=0}^{N-1}c_n^+|vac\rangle$$

$$\Rightarrow K_N(x,y)=\langle 0|\hat{\Psi}^+(x)\hat{\Psi}(y)|0\rangle=\sum_{n=0}^{N-1}\varphi_n(x)\varphi_n(y) \quad !$$

$\langle 0|c_n^+c_m|0\rangle=\delta_{nm}\,\Theta(N-1-m)$

Fermion density:

$$\rho(x)=\langle 0|\Psi^+(x)\Psi(x)|0\rangle=K_N(x,x)$$

$$\Leftrightarrow \rho(E)=K_N(E,E)=\sum_{n=0}^{N-1}\varphi_n^2(E)\qquad \left(=\int\sum_i\delta(E-E_i)\,P(\{E_n\})\,d^NE\right)$$

average density of levels (or fermions) at E

Level correlations $\Leftrightarrow$ density-density correlator!

Define $\rho(E) := \sum_i \delta(E-E_i)$

$$R_N(E,E') := \frac{\langle \rho(E)\rho(E')\rangle}{\langle \rho(E)\rangle\langle\rho(E)\rangle} \leftarrow \langle 0|\hat{\psi}^\dagger(E)\hat{\psi}(E)\hat{\psi}^\dagger(E')\hat{\psi}(E')|0\rangle$$

Wick's theorem for free fermions $\Rightarrow$ $\boxed{K_N(E,E)K_N(E',E') - K_N^2(E,E')}$

$(+\rho(E)\,\delta(E-E'))$

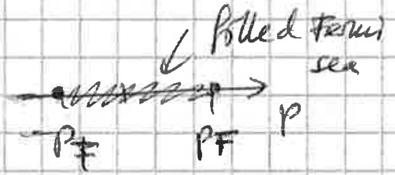
(This generalizes easily to multi-point correlators)

$\Rightarrow$ Find now $K_N(E,E')$ for large N! ($\Rightarrow$ Starting point to calculate the Tracy-Widom formula for largest eigenvalue!)

3.3. Level density

At large N: Fermi wavelength is short compared to scale on which potential $V(E)$ varies $\Rightarrow$ Use Thomas-Fermi approximation:

1d Fermi sea has a density $\rho = \frac{2p_F}{2\pi}$



Last fermion lives at energy $E_{N-1} = N-1+\frac{1}{2}$

Local depth of the Fermi sea: $E_{N-1} - \frac{x^2}{2}$ ← potential $= E_{kin}(x)$

$$\Rightarrow p_F^{(N)}(x) = \sqrt{2E_{kin}(x)} = \sqrt{2N-1-x^2}$$

$$\boxed{\Rightarrow \rho(x) = \frac{1}{\pi}p_F^{(N)}(x) = \frac{1}{\pi}\sqrt{2N-1-x^2}}$$ Wigner semi-circle!

3.4. Density-Density or two-level correlator

Here we really need K_N, and thus $\varphi_N(x)$ for large N.

$\Rightarrow$ Use semi-classical WKB approximation!

For $x^2 < 2N$ (classically allowed region at energy N)

$$\varphi_N(x) \propto 1/\sqrt{p_F^{(N)}(x)} \cdot \begin{cases} \cos\left[\int_0^x p_F^{(N)}(x')\,dx'\right] & N \text{ even} \\ \sin\left[\int_0^x p_F^{(N)}(x')\,dx'\right] & N \text{ odd}\end{cases}$$

Large N, x close to origin (small E in the middle of the spectrum):

$$p_F^{(N)}(x) \approx \sqrt{2N} \quad ; \quad \int_0^x p_F^{(N)}(x')\,dx' \approx \sqrt{2N}\, x$$

$$\Rightarrow K_N(x,y) = \text{const.} \cdot \frac{\sin\left(\sqrt{2N}(x-y)\right)}{x-y} \qquad \text{[Chr.-Darboux]}$$

$$\rho(x) = K_N(x,x) \approx \frac{\sqrt{2N}}{\pi} \Rightarrow \text{const.} = \frac{1}{\pi}$$

Mean level spacing: $\Delta = \frac{1}{\rho(x \approx 0)} \approx \frac{\pi}{\sqrt{2N}}$

sine kernel

$$\boxed{K_N(E,E') = \frac{\sqrt{2N}}{\pi} \frac{\sin\left(\sqrt{2N}\,(E-E')\right)}{\sqrt{2N}\,(E-E')} \Downarrow \Rightarrow \rho_0 \cdot \frac{\sin(s\pi)}{s\pi} \quad \text{as } N \to \infty}$$

$$(E-E')\sqrt{2N} = \frac{E-E'}{\Delta}\,\pi \equiv s \cdot \pi$$

$$\Rightarrow R_N(E,E') \xrightarrow[N\to\infty]{\delta(s)} + \frac{\rho_0^2 \left(1 - \left(\frac{\sin(s\pi)}{s\pi}\right)^2\right)}{\rho_0^2}$$

$K(E,E)\,K(E',E')$; $K(E,E')^2$

$$= \delta(s) + 1 - \frac{\sin^2(\pi s)}{(\pi s)^2}$$

$$= \underbrace{\delta(s) + 1 - \frac{1}{2\pi^2 s^2} + \frac{\cos 2\pi s}{\pi^2 s^2}}_{\equiv R_\infty(s)}$$

Properties:

- $s \ll 1$: $1 - \left(\frac{\sin(\pi s)}{\pi s}\right)^2 \approx \frac{\pi^2}{3} s^2$

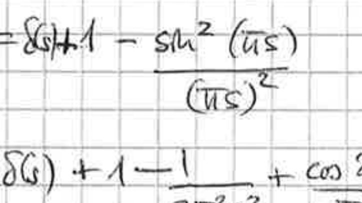
  ($\Leftrightarrow$ Wigner surmise: $\frac{32}{\pi^2} s^2$)

- Correlation hole: Around a given level E, there is a depletion of other levels!

$$\int_{-\infty}^{\infty} ds \left[R_{N\to\infty}(s) - \delta(s) - 1\right]$$

$R_{N\to\infty}(s)$: density of other levels you find at distance s

1: what you would find if you didn't condition on an existing level



$$= \int_{-\infty}^{\infty} ds \left(-\frac{\sin^2(\pi s)}{(\pi s)^2}\right) = -1 \leftarrow \text{exactly one level less.}$$

The fixed level repels the others and creates a full correlation hole

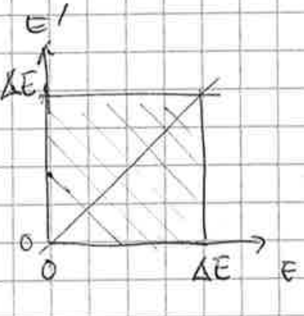
(exactly like fermions do!)

• The number of levels in a large interval $\Delta E \gg \Delta$ is very small

⇔ "incompressibility of levels":

$$n := \int_{-\Delta E/2}^{\Delta E/2} dE\, \rho(E) = \int_{\Delta E} dE\, \rho(E)$$

Fluctuations: $\langle n^2\rangle - \langle n\rangle^2 \equiv \Sigma(n) = \int_{\Delta E}\rho(E)\,dE\int_{\Delta E}\rho(E')\,dE'\,\left[R_N(E,E')-1\right]$



$$\simeq \int_{-\Delta E/2}^{\Delta E/2} d(E-E')\,\rho_0^2 \int_{|E-E'|/2}^{\Delta E-|E-E'|/2} d\left(\frac{E+E'}{2}\right)\left(R_\infty(s)-1\right)$$

$$\boxed{\langle n\rangle = \bar{n} = \rho_0\,\Delta E}$$

$$= \int_{-\bar{n}}^{+\bar{n}} ds\,\left(\bar{n}-|s|\right)\cdot\left(R_\infty(s)-1\right)$$

$$= \underbrace{\bar{n}\int_{-\infty}^{\infty}\left(R_\infty(s)-1\right)}_{=0} - \underbrace{\underbrace{2\bar{n}\int_{\bar{n}}^{\infty}\left(R_\infty(s)-1\right)}_{O(1/\bar{n})}}_{O(1)} - \underbrace{2\int_0^{\bar{n}} s\left(R_\infty(s)-1\right)ds}_{+\,2\int_0^{\bar{n}} s\,\frac{\sin^2\pi s}{\pi^2 s^2}\,ds}$$

$\sin^2\pi s = 1/2 +$ oscillatory

$$\simeq \frac{1}{\pi^2}\log(\bar{n}) + \ldots$$

⇒ Fluctuations $\langle n^2\rangle - \langle n\rangle^2$ only grow as $\sim \log(\langle n\rangle)$!

⇒ very rigid (incompressible)! (like neutron stars!)

(⇔ Poissonian: $\propto \langle n\rangle$!)

• Oscillatory term in $R_\infty(s)$ has period of mean level spacing Δ ⇒ It is as if there a crystalline structure that is remembered.

Origin is similar to Friedel oscillations at a hard wall

Oscillations reflect the Fermi wavevector and the abrupt jump of occupation of states from 1 to 0.

In the classical Coulomb plasma analogy one has a repulsive set of particles that tend to form a crystal (density wave). But Mermin-Wagner's theorem forbids true long range order ⇒

$\Rightarrow$ Oscillations die out at large s, but slowly $\sim \frac{\cos(2\pi s)}{s^2}$

- The non-oscillatory decay $R_\infty(s) - 1 = -\frac{1}{2\pi^2 s^2}$

  can already be obtained from the continuous mean field Coulomb analogy: and holds for general β:

$$\langle \rho(E)\, \rho(E') \rangle - \langle \rho(E) \rangle \langle \rho(E') \rangle \simeq \rho_0^2 \left(R_\infty(E-E') - 1 \right)$$

$$\overset{!}{=} \; -\frac{\delta \rho(E)}{\delta V(E')} \quad \text{from } \mathcal{L}[\rho]$$

From Coulomb minimum:

$$\rho(E) = \frac{1}{\pi^2} \sqrt{D^2 - E^2} \int_{-D}^{D} \frac{\frac{1}{\beta}(dV/dE')\; dE'}{\sqrt{D^2 - E'^2}\;\; E' - E}$$

$$\Rightarrow -\frac{\delta \rho(E)}{\delta V(E')} = \frac{1}{\pi^2} \sqrt{D^2 - E^2}\, \frac{1}{\beta} \frac{d}{dE'} \left(\frac{1}{\sqrt{D^2 - E'^2}} \frac{1}{E' - E} \right)$$

$$\underset{E \ll D}{\approx} -\frac{1}{\pi^2} \frac{1}{\beta} \frac{1}{(E'-E)^2}$$

$$\Rightarrow R_\infty(E - E') - 1 \simeq -\frac{1}{\rho_0^2 (E'-E)^2} \frac{1}{\beta \pi^2} = -\frac{1}{\beta \pi^2 s^2}$$

the stronger β

the stronger the repulsion $\Rightarrow$ faster decay

$$\Rightarrow \left[\langle n^2 \rangle - \langle n \rangle \right]_{n \text{ in } dE} \simeq \frac{1}{\beta \pi^2} \log(\bar{n})$$

$\Rightarrow$ smaller fluctuations

But: oscillations $\sim \frac{\cos 2\pi s}{s^{\beta}}$! (needs more complex application of orthogonal polynomials)

$\Rightarrow$ slower decay for larger β.