CFT exercises week 7

Exercise 1 ***Correlation functions from embedding formalism***

Using the embedding space formalism, compute the two-point function of a spin 2 primary:

$$\langle O_{AB}(P_1) O_{CD}(P_2)\rangle \, . \tag{1}$$

Then project the result to the d-dimensional physical space.

Hint: *the building blocks are* η_{AB}, P_1, P_2. *It is useful to begin by constructing a transverse matrix* W_{AB}.

Exercise 2 ***Conformal Blocks from Casimir differential equation***

In the embedding formalism, each primary operator is promoted to an homogeneous field on the future light-cone of the origin of $\mathbb{M}^{d+2}$,

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta}\mathcal{O}(P) \ , \qquad P^2 = 0, \quad \lambda > 0 \ . \tag{2}$$

In this formalism, conformal transformations are just $SO(d+1,1)$ *Lorentz transformations of Minkowski space* $\mathbb{M}^{d+2}$. *The conformal block decomposition can then be written as*

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\mathcal{O}_4(P_4)\rangle = \sum_k C_{12k} C_{k34}\, G^{(12)(34)}_{\Delta_k, l_k}(P_1,\ldots,P_4) \tag{3}$$

where

$$G^{(12)(34)}_{\Delta,l}(P_1,\ldots,P_4) = \frac{1}{P_{12}^{(\Delta_1+\Delta_2)/2} P_{34}^{(\Delta_3+\Delta_4)/2}} \left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}} g_{\Delta,l}(u,v) \ , \tag{4}$$

$P_{ij} = -2 P_i \cdot P_j$ *and* u, v *are conformal invariant cross ratios*

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}} \ , \qquad v = \frac{P_{14}P_{23}}{P_{13}P_{24}} \ . \tag{5}$$

The conformal blocks are eigenfunctions of the conformal Casimir,

$$\frac{1}{2}(J_{1,AB}+J_{2,AB})(J_1^{AB}+J_2^{AB}) G^{(12)(34)}_{\Delta,l}(P_1,\ldots,P_4) = \mathcal{C}_{\Delta,l}\, G^{(12)(34)}_{\Delta,l}(P_1,\ldots,P_4) \ , \tag{6}$$

with eigenvalue $\mathcal{C}_{\Delta,l} = \Delta(\Delta-d) + l(l+d-2)$, *where*

$$J_{AB} = i\left(P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A}\right) \tag{7}$$

are the Lorentz generators in $\mathbb{M}^{d+2}$ with indices $A, B = 0, 1, \ldots, d+1$.

***a**.* Show (using* `Mathematica`*) that (6) together with (4) is equivalent to*

$$\mathcal{D}\, g_{\Delta,l}(u,v) = \frac{1}{2}\mathcal{C}_{\Delta,l}\, g_{\Delta,l}(u,v) \tag{8}$$

where

$$\mathcal{D} =(1-u-v)\frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}+a+b\right)+u\frac{\partial}{\partial u}\left(2u\frac{\partial}{\partial u}-d\right) \tag{9}$$

$$-(1+u-v)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+a\right)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+b\right) \tag{10}$$

and $a = (\Delta_2 - \Delta_1)/2$ and $b = (\Delta_3 - \Delta_4)/2$.

***b**. It is convenient to parametrize the cross ratios by*

$$u = z\bar{z}\ , \qquad\qquad v = (1-z)(1-\bar{z})\ , \tag{11}$$

where z and $\bar{z}$ are independent variables. Show that for the choice $x_4 \to \infty$ and $x_{13}^2 = 1$ in Euclidean space, we have $z = |z|e^{i\theta}$ and $\bar{z} = |z|e^{-i\theta}$ with $|z|^2 = x_{12}^2$ and θ the angle between the vectors x_{12} and x_{13}.

***c**. Transform to the coordinates z and $\bar{z}$ defined in (11) and obtain*

$$\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}} + (d-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\frac{\partial}{\partial z}-(1-\bar{z})\frac{\partial}{\partial \bar{z}}\right) \tag{12}$$

with

$$\mathcal{D}_z = z^2(1-z)\frac{\partial^2}{\partial z^2} - (a+b+1)z^2\frac{\partial}{\partial z} - abz\ . \tag{13}$$

***d**. The small $|z|$ behaviour of the conformal block can be obtained from the leading order OPE. This was derived in class:*

$$g_{\Delta,l} \approx= \frac{l!}{2^l(h-1)_l}|z|^{\Delta}C_l^{h-1}(\cos\theta) \tag{14}$$

where $C_l^{h-1}(\cos\theta)$ is the Gegenbauer polynomial. Notice that this limit is particularly simple in two and four dimensions

$$g_{\Delta,l} \approx \frac{1}{2^l}|z|^{\Delta}\frac{e^{il\theta}+e^{-il\theta}}{1+\delta_{l,0}}\ , \qquad\qquad d=2\ , \tag{15}$$

$$g_{\Delta,l} \approx \frac{1}{2^l}|z|^{\Delta}\frac{e^{i(l+1)\theta}-e^{-i(l+1)\theta}}{e^{i\theta}-e^{-i\theta}}\ , \qquad\qquad d=4\ . \tag{16}$$

In two dimensions, the partial differential equation separates in two ordinary differential equations. Show that

$$g_{\Delta,l} = \frac{k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + k_{\Delta+l}(\bar{z})k_{\Delta-l}(z)}{2^l(1+\delta_{l,0})} \tag{17}$$

satisfies the boundary condition (15) if $k_\beta(z) \approx z^{\beta/2}$ *for small* z*, and the Casimir differential equation if*

$$\mathcal{D}_z k_\beta(z) = \frac{\beta}{2}\left(\frac{\beta}{2} - 1\right) k_\beta(z) \ . \tag{18}$$

Conclude that

$$k_\beta(z) = z^{\beta/2} \ {}_2F_1\left(\frac{\beta}{2} + a, \frac{\beta}{2} + b, \beta, z\right) \ . \tag{19}$$

e*. Check that*

$$g_{\Delta,l} = \frac{z\bar{z}}{2^l(z-\bar{z})}\big(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta+l}(\bar{z})k_{\Delta-l-2}(z)\big) \tag{20}$$

satisfies both the differential equation and the boundary condition in $d = 4$*.*