CFT exercises week 6

Exercise 1 ***Operator Product Expansion - scalar case***

The general form of the OPE of two scalar operators is

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}} \left[F^{(12k)}_{a_1\dots a_l}(x,\partial_y)\,\mathcal{O}_k^{a_1\dots a_l}(y)\right]_{y=0} \tag{1}$$

where the sum runs over all primary operators $\mathcal{O}_k$ *with spin* l *and dimension* Δ.

a*. Show that scale invariance implies that*

$$F^{(12k)}_{a_1\dots a_l}\left(\lambda x, \lambda^{-1}\partial_y\right) = \lambda^l F^{(12k)}_{a_1\dots a_l}(x,\partial_y) \tag{2}$$

b*. Compute the three-point function of scalar primary operators,*

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_3(w)\rangle = \frac{C_{123}}{|x|^{\Delta_1+\Delta_2-\Delta_3}|w|^{\Delta_3+\Delta_2-\Delta_1}|x-w|^{\Delta_1+\Delta_3-\Delta_2}}\ , \tag{3}$$

using the OPE above, and derive

$$\left[F^{(123)}(x,\partial_y)\left(1+\frac{y^2-2y\cdot w}{w^2}\right)^{-\Delta_3}\right]_{y=0} = \left(1+\frac{x^2-2x\cdot w}{w^2}\right)^{\frac{\Delta_2-\Delta_1-\Delta_3}{2}}\ . \tag{4}$$

c*.* Write a* `Mathematica` *program that uses the last equation to compute the coefficients* $a_{n,m}$ *for* $n+2m\leq 10$ *in the derivative expansion*

$$F^{(123)}(x,\partial_y) = \sum_{n,m=0}^{\infty} a_{n,m}(x\cdot\partial_y)^n(x^2\partial_y^2)^m \tag{5}$$

Suggestion: choose $w^2=1$ *in equation (4).*

d*.* Make a table of your results and try to guess an analytic formula for* $a_{n,m}$*. The function*

$$\texttt{Pochhammer[t,k]} = (t)_k = \frac{\Gamma(t+k)}{\Gamma(t)} = t(t+1)\dots(t+k-1) \tag{6}$$

will be very useful.

Exercise 2 ***Radial quantization of a free scalar field.***

1 *Consider the action for a free massless scalar field in Euclidean signature:*

$$S_E = \int d^d x \frac{1}{2}(\partial_\mu \phi)^2, \qquad g_{\mu\nu} = \delta_{\mu\nu}. \tag{7}$$

We want to quantize the theory on a surface of constant radius $r = (x^\mu x_\mu)^{1/2}$. *Different "constant time" surfaces are related by the scale transformation* $r \to e^\lambda r$. *If we define* $\sigma = \log r$, *time evolution is realized by the shift:* $\sigma \to \sigma + \lambda$. σ *will be our time variable.*

Change coordinates in the action to (σ, θ^i), *where* θ^i *collectively denotes the* $d-1$ *angular variables in spherical coordinates - for instance, when* $d = 3$, $\theta^i = (\theta, \phi)$. *Show that the action becomes*

$$S_E = \int d\sigma \int d\Omega \, \frac{1}{2}\left((\partial_\sigma \chi)^2 + \partial_i \chi \partial^i \chi + \left(\frac{d-2}{2}\right)^2 \chi^2 \right), \qquad \chi = e^{\frac{d-2}{2}\sigma}\phi, \tag{8}$$

where $d\Omega$ *is the area element on a sphere of radius 1 (for instance, when* $d = 3$, $d\Omega = \sin\theta d\theta d\phi$*). Disregard, as usual, total derivatives.*

2 *The usual Legendre transform almost gives the following Hamiltonian:*

$$H = \int d\Omega \, \frac{1}{2}\left(-(\partial_\sigma \chi)^2 + \partial_i \chi \partial^i \chi + \left(\frac{d-2}{2}\right)^2 \chi^2 \right). \tag{9}$$

Actually, the procedure yields $-H$, *which has a negative spectrum. One way to understand the minus sign is to think about the Wick rotation from Lorentzian signature (we can discuss this in class). Find a complete set of solutions of the equation of motion. For simplicity, work in* $d = 3$. *Recall that the spherical harmonics* $Y_{lm}(\theta, \phi)$ *form a complete set of solutions of the Laplace equation on a sphere. They have the following properties:*

$$\Box_{S^2} Y_{lm} = \left(\frac{1}{\sin\theta}\partial_\theta \sin\theta \partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 \right) Y_{lm} = -l(l+1) Y_{lm}, \tag{10}$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta, \phi) Y_{lm}(\theta', \phi') = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{\sin\theta}, \tag{11}$$

$$\int d\Omega \, Y_{lm}(\theta, \phi) Y^*_{l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'}, \tag{12}$$

$$Y^*_{lm} = (-1)^m Y_{l,-m}. \tag{13}$$

In particular, start with the following ansatz:

$$\chi(\sigma, \theta, \phi) = \sum_{lm} b_{lm}(\sigma) Y_{lm}(\theta, \phi). \tag{14}$$

Show that it is possible to expand the solution in terms of two sets of constant modes, a^+_{lm} *and* a^-_{lm}*, such that*

$$H = \sum_{lm} \frac{1}{2} \omega (a^-_{lm} a^+_{lm} + a^+_{lm} a^-_{lm}), \quad \omega = l + \frac{1}{2}. \tag{15}$$

In defining the modes, you should keep in mind that χ *is* not *a real field. Instead, you should impose the following hermiticity property:*

$$\chi^\dagger(\tau) = \chi(-\tau). \tag{16}$$

3 *The quantization of the theory follows from imposing the canonical commutation relations:*

$$[\chi(\sigma, \theta, \phi), \partial_\sigma \chi(\sigma, \theta', \phi')] = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{\sin\theta}. \tag{17}$$

The denominator on the right hand side is present in order to compensate the factor $\sin\theta$ *in the differential* $d\Omega$*. The plus sign, as opposed to the usual factor* i*, is related to the Wick rotation: defining a real time variable* $t = -i\sigma$ *one recovers the missing imaginary unit. Plug the mode expansion in the commutation relations and deduce that the energy spectrum is labeled by* n*-tuples of positive integers* $(l_1, \ldots, l_n)$*,* $n = 0, 1, 2, \ldots$ *:*

$$E_{l_1,\ldots l_n} = \left(l_1 + \frac{1}{2}\right) + \cdots + \left(l_n + \frac{1}{2}\right). \tag{18}$$

4 *Compare the spectrum with the scaling dimension of the local operators in the theory. Do they match? The state operator correspondence is the statement that the answer to this question is affirmative.*

5 *In the usual quantization on constant time slices, the Hamiltonian can be recovered as the integral of* T_{00} *on a fixed time-slice. Consider now the current* $j_\mu = -T_{\mu\nu}x^\nu$*, where* $T_{\mu\nu}$ *is the traceless stress-tensor. The expression for the latter in arbitrary dimension is as follows:*

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - \xi\left(\partial_\mu\partial_\nu - g_{\mu\nu}\Box\right)\phi^2, \quad \xi = \frac{d-2}{4(d-1)}. \tag{19}$$

Compare the Hamiltonian (9) *with the flux of the current on a sphere:*

$$\int d\Omega \, r^{d-1} \, \frac{x^\mu}{r} \, j_\mu. \tag{20}$$

If you want, you can keep the parameter ξ generic until the end, and check that this operator and the Hamiltonian have different spectra, unless the stress-tensor is chosen to be traceless.

Exercise 3 *The two point function of a scalar primary operator can be written as an inner product*

$$\langle \mathcal{O}(-\tau_1, \boldsymbol{x}_1)\mathcal{O}(\tau_2, \boldsymbol{x}_2)\rangle = \langle \mathcal{O}(\tau_1, \boldsymbol{x}_1)|\mathcal{O}(\tau_2, \boldsymbol{x}_2)\rangle \,, \tag{21}$$

where

$$|\mathcal{O}(\tau, \boldsymbol{x})\rangle = \mathcal{O}(\tau, \boldsymbol{x})|0\rangle \,, \qquad \tau < 0\,. \tag{22}$$

In the quantization with constant $x^1 = \tau$ surfaces, it is natural to decompose the state $|\mathcal{O}(\tau, \boldsymbol{x})\rangle$ into an eigenbasis of momenta $\hat{P}^\mu$. Notice that the conjugation rule implies that P^1 is hermitian and $\hat{P}^j$ for $j = 2, \ldots, d$ is anti-hermitian. In fact, $\hat{P}^1 = H$ is the hamiltonian and $\hat{P}^j_L = -i\hat{P}^j$ is the hermitian operator representing spatial momentum in the Lorentzian theory. We can then write

$$|\mathcal{O}(\tau, \boldsymbol{x})\rangle = \sum_\alpha \int \frac{E d\boldsymbol{k}}{(2\pi)^d} \, \psi_\alpha(\tau, \boldsymbol{x}; E, \boldsymbol{k})|E, \boldsymbol{k}, \alpha\rangle \,, \tag{23}$$

where $|E, \boldsymbol{k}, \alpha\rangle$ is an eigenstate of momentum

$$\hat{H}|E, \boldsymbol{k}, \alpha\rangle = E|E, \boldsymbol{k}, \alpha\rangle \,, \qquad \hat{P}^j|E, \boldsymbol{k}, \alpha\rangle = ik^j|E, \boldsymbol{k}, \alpha\rangle \,, \tag{24}$$

and the label α distinguishes states with the same momentum eigenvalue. Use $\left[\hat{P}^\mu, \hat{\mathcal{O}}(\tau, \boldsymbol{x})\right] = \partial_\mu \hat{\mathcal{O}}(\tau, \boldsymbol{x})$ to show that

$$\psi_\alpha(\tau, \boldsymbol{x}; E, \boldsymbol{k}) = e^{\tau E + i\boldsymbol{k}\cdot\boldsymbol{x}} q_\alpha(E, \boldsymbol{k})\,. \tag{25}$$

Using the normalization [1] $\langle E, \boldsymbol{k}, \alpha|E', \boldsymbol{k}', \alpha'\rangle = (2\pi)^d \delta_{\alpha\alpha'} \delta^{d-1}(\boldsymbol{k} - \boldsymbol{k}')\delta(E - E')$*, show that the two point function becomes*

$$\langle \mathcal{O}(-\tau_1, \boldsymbol{x}_1)\mathcal{O}(\tau_2, \boldsymbol{x}_2)\rangle = \frac{1}{(\tau^2 + \boldsymbol{x}^2)^\Delta} = \int \frac{dEd\boldsymbol{k}}{(2\pi)^d} e^{-E\tau + i\boldsymbol{k}\cdot\boldsymbol{x}} \rho(E, \boldsymbol{k}) \,, \qquad \tau > 0\,, \tag{26}$$

[1] If the normalization includes a positive factor depending on E and $\mathbf{k}$, that will not change the positivity properties of the spectral density.

where $\tau = -(\tau_1 + \tau_2)$*,* $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ *and the spectral density is* $\rho(E, \boldsymbol{k}) = \sum_\alpha |q_\alpha(E, \boldsymbol{k})|^2$*. Show that*

$$\rho(E, \boldsymbol{k}) = \frac{2\pi^{\frac{d}{2}+1}}{\Gamma(\Delta)\Gamma\left(\Delta - \frac{d}{2} + 1\right)} \Theta\left(E - |\boldsymbol{k}|\right) \left(\frac{E^2 - \boldsymbol{k}^2}{4}\right)^{\Delta - \frac{d}{2}} . \qquad (27)$$

Conclude that there is no particle interpretation of this spectral density for generic Δ*. Conclude also that reflection positivity (or unitarity) implies that* $\Delta \geq \frac{d}{2} - 1$*. Give a particle interpretation to the spectral density for* $\Delta = \frac{d}{2} - 1$ *and* $\Delta = d - 2$*.*