CFT exercises, week 5

Exercise 1 *The generators of the conformal algebra can be represented as follows*

$$\hat{P}_\mu = -i\partial_\mu \ , \qquad \hat{L}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad (1)$$
$$\hat{D} = -\, x\cdot\partial \ , \qquad \hat{K}_\mu = i(2x_\mu\, x\cdot\partial - x^2\partial_\mu)$$

a) *In a unitary representation, there is a positive definite inner product such that*

$$\hat{D}^\dagger = \hat{D} \ , \qquad \hat{K}_\mu^\dagger = \hat{P}_\mu \ , \qquad \hat{L}_{\mu\nu}^\dagger = \hat{L}_{\mu\nu} \ . \qquad (2)$$

Show that unitarity implies that the dimension (or eigenvalue of $\hat{D}$) of a scalar primary state $|\mathcal{O}\rangle$ can not be lower than $\frac{d-2}{2}$, and that the bound is saturated by a state created by a free massless scalar field (obeying the equation of motion $\partial^2\mathcal{O}(x) = 0$).

b) *Show that a vector primary state $|\mathcal{O}^\alpha\rangle$ contained in a unitary representation must have dimension larger or equal to $d-1$. Show that when the bound is saturated, the state is created by a conserved current. Recall that for a spin 1 state,*

$$\hat{L}_{\mu\nu}|\mathcal{O}^\alpha\rangle = (M_{\mu\nu})^\alpha{}_\beta|\mathcal{O}^\beta\rangle \ , \qquad (M_{\mu\nu})^\alpha{}_\beta = i(\eta_{\nu\beta}\delta^\alpha_\mu - \eta_{\mu\beta}\delta^\alpha_\nu) \ . \qquad (3)$$

Hint: Compute the norm of $P_\mu|\mathcal{O}^\mu\rangle$.

c) *Verify that the operator*

$$\hat{C} = \hat{D}^2 - \frac{1}{2}\left(\hat{K}_\mu\hat{P}^\mu + \hat{P}_\mu\hat{K}^\mu\right) + \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}^{\mu\nu} \qquad (4)$$

is a Casimir of the conformal algebra (i.e. it commutes with all its generators). Determine its value for a scalar and a vector primary state.

d) *Generalize questions **b**) and **c**) for symmetric traceless primary states $|\mathcal{O}^{\alpha_1\dots\alpha_l}\rangle$. Recall that for spin l states,*

$$\hat{L}_{\mu\nu}|\mathcal{O}^{\alpha_1\dots\alpha_l}\rangle = \sum_{i=1}^{l}(M_{\mu\nu})^{\alpha_i}{}_\beta|\mathcal{O}^{\alpha_1\dots\alpha_{i-1}\beta\alpha_{i+1}\dots\alpha_l}\rangle \ . \qquad (5)$$

You should find that the dimension of such a state in a unitary theory must be greater or equal to $d-2+l$.

Exercise 2 ***Analytic structure of the two-point function***

Correlation functions in a Euclidean QFT are always *time ordered (we write T_E here to emphasize this is Euclidean time ordering):*

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \langle 0|T_E\left\{\hat{\mathcal{O}}_1(\tau_1,\boldsymbol{x}_1)\dots\hat{\mathcal{O}}_n(\tau_n,\boldsymbol{x}_n)\right\}|0\rangle\,, \qquad (6)$$

The reason for this is that out-of-time ordered correlators do not exist in general in Euclidean signature:

a) *indeed, consider a two-point function of operators which are odered as written:*

$$\langle 0|\,\hat{\mathcal{O}}(\tau_1,\vec{x}_1)\hat{\mathcal{O}}(\tau_2,\vec{x}_2)\,|0\rangle\ . \qquad (7)$$

Use the time evolution equation for the operators:

$$\hat{\mathcal{O}}(\tau,\vec{x}) = e^{\tau\hat{H}}\hat{\mathcal{O}}(0,\vec{x})e^{-\tau\hat{H}}\,. \qquad (8)$$

In general H is only bounded from below, not from above. By using this fact, show that the two-point function is infinite if the operators are anti-time ordered. Correlation functions computed from a path-integral indeed compute Euclidean time ordered correlators.

b) *Consider the two point function of a primary operator $\mathcal{O}$ as a function of time, at fixed distance $\vec{x}$:*

$$F(\tau) = \langle \mathcal{O}(\tau,\vec{x})\mathcal{O}(0)\rangle = \frac{1}{(\vec{x}^2+\tau^2)^{\Delta}}. \qquad (9)$$

Define $F(\tau)$ for complex values of τ by analytic continuation, and find the singularities of $F(\tau)$ in the complex τ-plane, for generic Δ.

c) *Set*

$$\tau = it+\epsilon, \quad t,\,\epsilon\in\mathbb{R}. \qquad (10)$$

Then as $\epsilon\to 0$, you get a correlation function in Lorentzian time. Compute the vacuum expectation value of $[\hat{\mathcal{O}}(it,\vec{x}),\hat{\mathcal{O}}(0)]$. Is it compatible with causality? Hint*: give a small real part to $\tau = it+\epsilon$, with $\epsilon\to 0^{\pm}$. The ordering of operators depends on the sign of ϵ.*

Exercise 3 *Imagine that we want to check the predictions of conformal invariance on the lattice. We perform a Monte Carlo simulation of the 2d Ising model on a square lattice, and we want to measure the two-point function of the spin operator σ. In the critical system, in the continuum limit, σ is predicted to have the following scaling dimension:*

$$\Delta_\sigma = \frac{1}{8}\ . \qquad (11)$$

However, the lattice operator $S_{i,j}$ placed at the point (i,j) is in general a linear combination of all the continuum operators with the correct spacetime and internal symmetry properties:

$$S_{i,j} = c_1 a^{\Delta_\sigma}\sigma(i,j) + \sum_k c_k a^{\Delta_k} O_k(i,j) \ , \tag{12}$$

where a is the lattice spacing and the subleading operators O_k are to be determined.

a) *What are the symmetries preserved by the realization of the model on a 2d square lattice? Assume the lattice to be infinite (even if in real life numerical experiments finite size effects are important!).*

b) *The low lying spectrum of $\mathbb{Z}_2$ odd primaries*[1] *in the 2d Ising model is as follows:*

$$(\Delta, j) \ : \left(\frac{1}{8}, 0\right) \ , \ \left(\frac{25}{8}, 3\right) \ , \ \dots \tag{13}$$

Fix the linear combination (12) up to dimension $\Delta = 33/8 = 1/8 + 4$. Of course, the value of the independent coefficients c_i should be fixed by the simulation.

c) *What is the functional form of the two-point function of S? You will have to compute the two point function of a spinning primary in 2d. It is convenient to use complex coordinates $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$ to find it. You can then pick some order one coefficients, set the lattice spacing $a = 1$, and draw some plots in Mathematica to get an idea of the radial and angular dependence.*

d) *There is a second source of deviations from the conformal fixed point. After tuning the relevant couplings – critical temperature and vanishing magnetic field – the lattice Hamiltonian still differs from the critical action by infinitely many irrelevant deformations. Consider only the leading one:*

$$H_{\text{lat}} = H_* + g \int d^2x \, a^{\Delta_{\epsilon'}-2}\epsilon'(x) \ , \tag{14}$$

where ϵ' is a $\mathbb{Z}_2$ even scalar of dimension $\Delta_{\epsilon'} = 4$. Determine the exponents of the power law corrections to the two-point function of S at leading order in g.

[1] More precisely, in the 2d literature these operators are called quasi-primaries, because primaries in 2d are something else which we may or may not explain in class depending on time.