CFT exercises, week 3

Exercise 1 *Show that the generators obey the following commutation relations*

$$\begin{aligned}
[D, P_\mu] &= P_\mu \ , \qquad [D, K_\mu] = -K_\mu \ , \qquad [K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2i\, M_{\mu\nu} \ , \\
[M_{\mu\nu}, P_\alpha] &= i\left(\delta_{\mu\alpha} P_\nu - \delta_{\nu\alpha} P_\mu\right) \ , \qquad [M_{\mu\nu}, K_\alpha] = i\left(\delta_{\mu\alpha} K_\nu - \delta_{\nu\alpha} K_\mu\right) \ , \\
[M_{\alpha\beta}, M_{\mu\nu}] &= i\left(\delta_{\alpha\mu} M_{\beta\nu} + \delta_{\beta\nu} M_{\alpha\mu} - \delta_{\beta\mu} M_{\alpha\nu} - \delta_{\alpha\nu} M_{\beta\mu}\right) \ . \qquad (1)
\end{aligned}$$

Exercise 2 *We saw in class that*

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_4) \rangle = \frac{\mathcal{A}(u,v)}{\left(x_{12}^2 x_{34}^2\right)^{\Delta}} \ . \qquad (2)$$

Generalize (2) *for 4 different operators. Generalize it also for the case of a n-point function. How many independent cross-ratios are there in this case?*

Exercise 3 *The correlator* (2) *is invariant under permutations of the points x_i. Show that this implies*

$$\mathcal{A}(u,v) = \mathcal{A}(u/v, 1/v)\,, \qquad \mathcal{A}(u,v) = \left(\frac{u}{v}\right)^{\Delta} \mathcal{A}(v,u)\,. \qquad (3)$$

Exercise 4 ***Tensor primary fields - three point function***

A tensor primary field of scaling dimension Δ and spin J transforms as follows

$$T'_{\mu_1 \dots \mu_J}(x) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta - J}{d}} \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} \dots \frac{\partial x'^{\nu_J}}{\partial x^{\mu_J}} T_{\nu_1 \dots \nu_J}(x') \ . \qquad (4)$$

***a**. Verify that*

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) j^\mu(x_3) \rangle = C_{12j} \frac{V^\mu(x_1, x_2, x_3)}{|x_{12}|^{\Delta_1 + \Delta_2 - \Delta + 1} |x_{13}|^{\Delta_1 + \Delta - \Delta_2 - 1} |x_{23}|^{\Delta_2 + \Delta - \Delta_1 - 1}} \qquad (5)$$

has the correct transformation properties of a three point function of a vector and two scalar primary operators in a CFT. In this expression, Δ is the dimension of the vector operator j^μ, Δ_i is the dimension of the scalar operator $\mathcal{O}_i$, C_{12j} is a constant and

$$V^\mu(x_1, x_2, x_3) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} \ . \qquad (6)$$

Suggestion: start by showing that under inversion $x'^{\mu} = x^{\mu}/x^2$, we have

$$(x'_{ij})^2 = \frac{x_{ij}^2}{x_i^2 x_j^2} \ , \qquad V_{\mu}(x_1, x_2, x_3) = \left.\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right|_{x=x_3} V_{\nu}(x'_1, x'_2, x'_3) \ . \tag{7}$$

***b**. Similarly, verify that*

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)T^{\mu\nu}(x_3)\rangle = C_{12T}\frac{H^{\mu\nu}(x_1,x_2,x_3)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+2}|x_{13}|^{\Delta_1+\Delta-\Delta_2-2}|x_{23}|^{\Delta_2+\Delta-\Delta_1-2}} \tag{8}$$

transforms appropriately under conformal transformations with $T^{\mu\nu}$ a primary field of dimension Δ and spin 2 (symmetric traceless tensor). Here, the numerator is

$$H^{\mu\nu} = V^{\mu}V^{\nu} - \frac{1}{d}V_{\alpha}V^{\alpha}\delta^{\mu\nu} \ , \tag{9}$$

and you can use the identities (7) without proof.

***c**. Consider a free massless scalar field with Euclidean action*

$$S[\varphi] = \int d^d x \frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi \ . \tag{10}$$

Show that the two-point function is given by (assume $d > 2$)

$$\langle\varphi(x)\varphi(y)\rangle = \frac{\mathcal{N}}{|x-y|^{d-2}} \ , \qquad \mathcal{N} = \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}} \ . \tag{11}$$

Recall that the Γ-function is defined by

$$\Gamma(z) = \int_0^{\infty} dt\, t^{z-1}e^{-t} \ , \qquad \Re\, z > 0 \ . \tag{12}$$

***d**. In the same theory, compute the three point function*

$$\langle\varphi(x_1)\varphi(x_2)T^{\mu\nu}(x_3)\rangle \ , \tag{13}$$

where

$$T_{\mu\nu} =: \partial_{\mu}\varphi\partial_{\nu}\varphi : -\frac{1}{4(d-1)}\left((d-2)\partial_{\mu}\partial_{\nu} + \eta_{\mu\nu}\partial^2\right) : \varphi^2 : \tag{14}$$

is the stress-energy tensor. Compare your result with (8) and determine the dimension Δ of the stress-energy tensor, the dimension Δ_{φ} and the constant $C_{\varphi\varphi T}$.