## CFT exercises, week 2

**Exercise 1** *Consider an invertible redefinition* $\tilde{g}(g)$ *of a coupling constant* $g$*. Assume that* $g = 0$ *is a fixed point of the RG flow and that* $\tilde{g}(0) = 0$*. Write the beta function*

$$\tilde{\beta}(\tilde{g}) = \frac{d\tilde{g}}{d\log\Lambda} \tag{1}$$

*as a perturbative expansion in the coupling* $\tilde{g}$ *using the perturbative expansions*

$$\beta(g) = \frac{dg}{d\log\Lambda} = b_1 g + b_2 g^2 + \ldots \tag{2}$$

*and*

$$\tilde{g} = a_1 g + a_2 g^2 + \ldots \tag{3}$$

*If* $b_1 \neq 0$ *then show that one can choose* $a_i$*'s such that* $\tilde{\beta}$ *is linear to all orders in* $\tilde{g}$*. Is the result physically reasonable?*

**Exercise 2** ***Dangerously irrelevant operators***

*Consider the (singular part of the) free energy of the Ising model in the mean field approximation*

$$f(h,t,\lambda) = \min_M \left[ -hM + \frac{t}{2}M^2 + \frac{\lambda}{12}M^4 \right] , \tag{4}$$

*and verify that it satisfies the scaling law*

$$f(h,t,\lambda) = b^{-d} f\left(b^{y_h}h, b^{y_t}t, b^{y_\lambda}\lambda\right) \tag{5}$$

*with renormalization group eigenvalues*

$$y_h = \frac{d}{2} + 1 , \qquad y_t = 2 , \qquad y_\lambda = 4 - d . \tag{6}$$

*Recall that these are the RG eigenvalues of the Gaussian fixed point. Using the standard formulas derived in the lectures, one would conclude that the critical exponents are given by the second column in next table. However, these are different from the actual critical exponents associated with Mean Field Theory free energy (4), which are given in the third column.*

| *Exponent* | *Gaussian* | *MFT* |
|---|---|---|
| $\alpha$ | $2-\frac{d}{2}$ | *0* |
| $\beta$ | $\frac{d-2}{4}$ | $\frac{1}{2}$ |
| $\gamma$ | *1* | *1* |
| $\delta$ | $\frac{d+2}{d-2}$ | *3* |
| $\nu$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\eta$ | *0* | *0* |

*Go through the general argument presented in the lectures and find the assumption made that is not valid in this case. Hint: study the free energy when* $\lambda \to 0$ *and* $t < 0$.

**Exercise 3** *Consider the Hamiltonian*

$$H = -\frac{1}{2}\sum_{x,y}\hat{J}(x-y)s(x)s(y) - \mathcal{H}\sum_{x}s(x) + \lambda\sum_{x}\left(s(x)^2-1\right)^2 \tag{7}$$

*where* $x$ *and* $y$ *label sites in a* $d$*-dimensional hypercubic lattice and* $s(x) \in \mathbb{R}$. *The coupling* $\hat{J}(x-y)$ *is equal to* $J$ *if* $x$ *and* $y$ *are nearest neighbours and it vanishes otherwise. Argue that in the limit* $\lambda \to \infty$ *this hamiltonian reduces to the usual Ising model where* $s(x) = \pm 1$. *We shall assume that the Hamiltonian with finite* $\lambda$ *can also describe the Ising critical point. This is motivated by universality and the intuition from BSTs.*

*Show that in the continuum limit where the lattice spacing* $a \to 0$, *the Hamiltonian can be written as*

$$H = \int d^dx\left[\frac{1}{2}(\partial\phi)^2 + t\,a^{-2}\phi^2(x) + u\,a^{d-4}\phi^4(x) + h\,a^{-1-\frac{d}{2}}\phi(x) + \dots\right], \tag{8}$$

*where the field* $\phi(x) = \sqrt{J}a^{\frac{2-d}{2}}s(x)$ *and the dimensionless couplings are*

$$t = -\frac{2\lambda}{J} - d\,, \qquad u = \frac{\lambda}{J^2}\,, \qquad h = -\frac{\mathcal{H}}{\sqrt{J}}\,. \tag{9}$$

*The* $\dots$ *in* (8) *stand for terms with more than two derivatives. Use dimensional analysis to conclude that a generic term with* $p$ *derivatives and* $n$ *fields,*

$$g\,a^{-y_g}\int d^dx\,\phi\,\partial^p\phi^{n-1}\,, \tag{10}$$

*has* $y_g = d - p - n\frac{d-2}{2}$*. The fixed point* $t = u = h = 0$ *is called the Gaussian fixed point. Check that for* $d > 4$*,* $t$ *and* $h$ *are the only relevant couplings of the Gaussian fixed point. In fact the operator* $\phi^3(x)$ *is relevant. However, it is a* ***redundant*** *operator because it can be removed from the hamiltonian by a field redefinition* $\phi(x) \to \phi(x) + const$*. In general, there is an infinite number of redundant operators that can be removed by local field redefinitions* $\phi(x) \to \phi(x) + c_1\phi^3(x) + c_2\partial^2\phi(x) + \ldots$*.*

*The continuum limit of the partition function*

$$Z = \int \prod_x d\phi(x) e^{-H[\phi]} \tag{11}$$

*is equivalent to the path integral formulation of (Euclidean) QFT, with the Hamiltonian playing the role of the Euclidean action.*