CFT exercises, week 1

Exercise 1 *Consider the propagator of a massive scalar field in Euclidean space,*

$$G(x) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{i\,k\cdot x}}{k^2+m^2} \ . \tag{1}$$

Determine its large-distance behaviour (for $x \gg 1/m$). Hint: *Use the identity*

$$\frac{1}{A} = \int_0^\infty dt\, e^{-tA} \ , \tag{2}$$

to do the Fourier transform to position space and evaluate the t integral using the saddle point approximation.

Estimate the behaviour of the propagator in the opposite limit $x \ll 1/m$. How does it compare with the propagator of a massless field?

A possible definition of correlation length is

$$\xi^2 = \frac{\int d^dx\, x^2 G(x)}{\int d^dx\, G(x)} \tag{3}$$

Show that this gives

$$\xi^2 = -\frac{1}{\hat{G}(0)} \frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k^\mu} \hat{G}(k)\bigg|_{k=0} = \frac{2d}{m^2} \tag{4}$$

where $\hat{G}(k)$ is the propagator in momentum space and in the last step we used the form of the propagator of a free massive scalar field.

Exercise 2 *Show that the scaling form*

$$f_s(t,h) = \left|\frac{t}{t_0}\right|^{\frac{d}{y_t}} \Phi_\pm\left(\frac{h}{h_0}\left|\frac{t}{t_0}\right|^{-\frac{y_h}{y_t}}\right) \tag{5}$$

of the singular part of the free energy, predicts the following expressions for the critical exponents introduced in section 1.1.1 of the notes,

$$\alpha = 2 - \frac{d}{y_t}\,, \qquad \beta = \frac{d-y_h}{y_t}\,, \qquad \gamma = \frac{2y_h-d}{y_t}\,, \qquad \delta = \frac{y_h}{d-y_h}\,. \tag{6}$$

Check that these imply the following scaling relations,

$$\alpha + 2\beta + \gamma = 2\,, \qquad \alpha + \beta + \beta\delta = 2\,. \tag{7}$$

In general, the scaling relations can be more complicated if there are more relevant operators.

Exercise 3 *Argue that the correlation length ξ satisfies*

$$\xi(u_t, u_h, u_I, \dots) = b\,\xi\,(b^{y_t}u_t, b^{y_h}u_h, b^{y_I}u_I, \dots)\ . \tag{8}$$

Use this to derive the following critical behaviour of ξ at zero magnetic field,

$$\xi \sim |t|^{-\frac{1}{y_t}} \qquad \Rightarrow \qquad \nu = \frac{1}{y_t}\,. \tag{9}$$

Compare this behaviour with the one of the correlation length of the 1D Ising model, found in problem 1.2.1 of the notes, that is

$$\xi = \frac{const}{\log \tanh 1/t}\ . \tag{10}$$

Exercise 4 ***Mean Field Approximation***

Prove Feynman's inequality

$$Tr\, e^{-H} \geq Tr\, e^{-H' - \langle H-H' \rangle_{H'}}\ , \tag{11}$$

where $[H, H'] = 0$ and

$$\langle \mathcal{O} \rangle_{H'} = \frac{Tr\, \mathcal{O}\, e^{-H'}}{Tr\, e^{-H'}}\ . \tag{12}$$

Choosing the hamiltonian H' to maximize the right hand side of (11) is a systematic way to implement a mean-field approximation. Use the hamiltonian

$$H' = -h' \sum_x s(x) \tag{13}$$

to study the Ising model hamiltonian on a hyper-cubic lattice,

$$H = -J \sum_{<x,y>} s(x)s(y) - h \sum_x s(x) \tag{14}$$

in the mean-field approximation. Show that the free energy per spin in this approximation

$$f_{MF} = -\frac{1}{N} \log \max_{h'} Tr\, e^{-H' - \langle H-H' \rangle_{H'}}\ , \tag{15}$$

can be written as

$$f_{MF} = \min_M \left[-\log 2 - hM + \frac{1-Jz}{2} M^2 + \frac{1}{12} M^4 + \mathcal{O}(M^6) \right]\ , \tag{16}$$

where z denotes the number of nearest neighbours of each spin. What is the critical temperature? Plot the phase diagram and compute the thermodynamical critical exponents for the Ising model in d dimensions using this approximate free energy.

To determine the spin two-point correlation function we need to allow for space dependent magnetic fields $h(x)$ and $h'(x)$. By moving to Fourier space, determine the spin two-point function at zero magnetic field and read off the ν and η critical exponents in the mean-field approximation.