Solutions Problem Sheet 5: Quantum Channels - Stinespring, Choi States and Complete Positivity

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. Write down a Stinespring dilation unitary for:

   a) The completely dephasing single qubit channel (i.e. that channel that kills of all coherence in the computational basis).

   b) The channel the kills of all coherence in the X basis instead of the computational basis?

   c) The channel that prepares an arbitrary mixed state ρ? Describe the general procedure and implement explicitly for a single qubit state with Bloch vector $\boldsymbol{r} = (0.2, 0, 0.2)$.

   Answer:

   a) This corresponds to the unitary U given in $6.a$):

   $$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes X_B.$$

   b) Here we simply need to change the basis of the system by applying Hadamard (on the system):

   $$U = |+\rangle\langle +|_A \otimes \mathbb{I}_B + |-\rangle\langle -|_A \otimes X_B.$$

   We can check that the output state of the channel is $\mathcal{E}(\rho_A) = \langle +|\rho_A|+\rangle |+\rangle\langle +| + \langle -|\rho_A|-\rangle |-\rangle\langle -|$.

   c) Here we need to create the purification of an arbitrary mixed state $\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ i.e. we are looking for the unitary U such that $U|0_A, 0_B\rangle = |\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i}|\lambda_i\rangle_A|i\rangle_B$. Knowing the first column of the unitary, we can use Gram-Schmidt process to get the other columns (i.e. to get an orthonormal basis containing $|\psi_{AB}\rangle$). Here you may use mathematica to find the solution of the single qubit state.

   Another method more adapted to quantum computing can be used if we know how to initialize the $\{|\lambda_j\rangle\}$ and the state $|\phi\rangle = \sum_j \sqrt{\lambda_j}|j\rangle$. Let say we know unitaries U_j (acting on the system) such that $U_j|\mathbf{0}\rangle = |\lambda_j\rangle$ and W (acting on the environment) such that $W|\mathbf{0}\rangle = |w\rangle$, then we first initialize $|w\rangle$ in the ancilla register and then we applied controlled-U_j gates e.g. $\sum_j U_j \otimes |j\rangle\langle j|_B$ (if some $\lambda_j = 0$, then $U_j = \mathbb{1}_A$), so for example we may have $U = (\sum_j U_j \otimes |j\rangle\langle j|_B)(\mathbb{1}_A \otimes W)$.

   For a single qubit state we can use $V(\phi,\theta) = R_Z(\phi)R_Y(\theta)$ to initialize any single qubit state as $V(\phi,\theta)|0\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ with $R_\sigma(\alpha) = \exp(-i\sigma\alpha/2)$. In the case where the Bloch vector is $(0.2, 0, 0.2)$, we have the eigenvalues $\lambda_\pm = \frac{5\pm\sqrt{2}}{10}$ and the corresponding eigenvectors $|\lambda_+\rangle = \frac{1}{\sqrt{2\sqrt{2}}}\left(\frac{1}{\sqrt{\sqrt{2}-1}}|0\rangle + \sqrt{\sqrt{2}-1}|1\rangle\right)$ and $|\lambda_-\rangle = \frac{1}{\sqrt{2\sqrt{2}}}\left(\frac{1}{\sqrt{\sqrt{2}+1}}|0\rangle - \sqrt{\sqrt{2}+1}|1\rangle\right)$. So, we choose $W = V(0, \arccos(\sqrt{2}/5))$, $U_+ = V(0,\pi/4)$ and $U_- = V(\pi, 3\pi/4)$ such that $U = (\mathbb{1}_A \otimes X_B)C_B\text{-}U_-(\mathbb{1}_A \otimes X_B)C_B\text{-}U_+(\mathbb{1}_A \otimes W)$ where $C_B\text{-}U_i$ is the control-unitary gate with control on B and unitary on A.

2. Write $|\mathrm{vec}(A)\rangle$ for the set of Pauli matrices.

   Answer: For the Paulis we have $|\mathrm{vec}(X)\rangle = |01\rangle + |10\rangle$, $|\mathrm{vec}(Y)\rangle = -i|01\rangle + i|10\rangle$, $|\mathrm{vec}(Z)\rangle = |00\rangle - |11\rangle$ and $|\mathrm{vec}(\mathbb{1})\rangle = |00\rangle + |11\rangle)$ (notice that it corresponds to the 4 Bell states up to a normalization factor).

3. Prove the following vectorization inequalities:

   i. $|\text{vec}(A \odot B)\rangle = |\text{vec}(A)\rangle \odot |\text{vec}(B)\rangle$ where $\odot$ denotes the Hadamard product (i.e. entrywise multiplication).

   ii. $\text{Tr}[A^\dagger B] = |\text{vec}(A)\rangle^\dagger |\text{vec}(B)\rangle$

   Answer: Let us define $A = \sum_{i,j} a_{ij} |i\rangle\langle j|$ and $B = \sum_{i,j} b_{ij} |i\rangle\langle j|$.
   i. We have $A \odot B = \sum_{i,j} a_{ij} b_{ij} |i\rangle\langle j|$. So $|\text{vec}(A \odot B)\rangle = \sum_{i,j} a_{ij} b_{ij} |i\rangle \otimes |j\rangle$. On the other hand we have $|\text{vec}(A)\rangle \odot |\text{vec}(B)\rangle = (\sum_{i,j} a_{ij} |i\rangle \otimes |j\rangle) \odot (\sum_{i,j} b_{ij} |i\rangle \otimes |j\rangle) = \sum_{i,j} a_{ij} b_{ij} |i\rangle \otimes |j\rangle$.
   ii. Here the proof is also straightforward. $|\text{vec}(A)\rangle^\dagger |\text{vec}(B)\rangle = \langle \text{vec}(A) | \text{vec}(B)\rangle$
   $= (\sum_{i,j} a^*_{ij} \langle i| \otimes \langle j|)(\sum_{i,j} b_{ij} |i\rangle \otimes |j\rangle) = \sum_{i,j} a^*_{ij} b_{ij} = \sum_{i,j} (A^\dagger)_{ji} (B)_{ij} = \text{Tr}[A^\dagger B]$.

4. Use vectorization for prove that the maximally entangled state $\propto \sum_i |ii\rangle$ is invariant under $U \otimes U^*$.

   Answer:

   Here we use the property $|\text{vec}(ABC)\rangle = A \otimes C^T |\text{vec}(B)\rangle$ (you can prove it yourself as an exercise). This implies that $U \otimes U^* |\text{vec}(\mathbb{1})\rangle = |\text{vec}(U \mathbb{1} U^\dagger)\rangle = |\text{vec}(\mathbb{1})\rangle$.

5. Consider the Choi state $\frac{2}{3}|\phi_-\rangle\langle\phi_-| + \frac{4}{3}|\psi_+\rangle\langle\psi_+|$. Write down a minimal set of Kraus operators for this channel. Hence describe the action (i.e. inputs and outputs) of the channel.

   Answer: The corresponding minimal set of Kraus operators is $\{\frac{1}{\sqrt{3}} Z, \sqrt{\frac{2}{3}} X\}$. From a single qubit state input (uniquely) defined by its Bloch vector $\{x, y, z\}$, the output state after the channel has Bloch vector $\{x/3, -y, -z/3\}$.

6. Suppose (as in the problem sheet from last week) we have a qubit system A interacting with a qubit environment through the unitary

   $$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes X_B \tag{1}$$

   a) Write down the Choi state for the channel induced on the system assuming that the environment qubit starts in the state $|0\rangle$.

   b) Hence write down Kraus operators for the channel.

   c) What are the Kraus operators if instead

   $$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes e^{-i\lambda t X_B} \ ? \tag{2}$$

   Answer:

   We have $J(\mathcal{E}) = \mathcal{E} \otimes \mathbb{1}(|vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|) = \mathcal{E}(|0\rangle\langle 0|) \otimes |0\rangle\langle 0| + \mathcal{E}(|1\rangle\langle 1|) \otimes |1\rangle\langle 1| + \mathcal{E}(|0\rangle\langle 1|) \otimes |0\rangle\langle 1| + \mathcal{E}(|1\rangle\langle 0|) \otimes |1\rangle\langle 0|$. So, knowing the action of the channel on each coefficient of the density matrix directly gives the Choi state. Then we can deduce the Kraus operators A_i as $J(\mathcal{E}) = \sum_i |vec(A_i)\rangle\langle vec(A_i)|$.

   a) The Choi state is $J(\mathcal{E}) = |00\rangle\langle 00| + |11\rangle\langle 11|$.

   b) Hence Kraus operators $\{|0\rangle\langle 0|_A, |1\rangle\langle 1|_A\}$ or $\{\frac{1}{\sqrt{2}} \mathbb{1}_A, \frac{1}{\sqrt{2}} Z_A\}$ can be used to realise this channel.

   c) The effect of this channel is to multiply the off-diagonal element of the density matrix by $\cos(\lambda t)$ and let the diagonal unchanged, so the Choi state is $J(\mathcal{E}) = |11\rangle\langle 11| + |00\rangle\langle 00| + \cos(\lambda t)|00\rangle\langle 11| + \cos(\lambda t)|11\rangle\langle 00|$. So, we can easily deduce Kraus operators $\{\sqrt{\cos(\lambda t)}\mathbb{1}, \sqrt{1 - \cos(\lambda t)}|0\rangle\langle 0|_A, \sqrt{1 - \cos(\lambda t)}|1\rangle\langle 1|_A\}$ or also $\{\sqrt{\frac{1+\cos(\lambda t)}{2}} \mathbb{1}_A, \sqrt{\frac{1-\cos(\lambda t)}{2}} Z_A\}$ for example.