Solutions Problem Sheet 12: Pure state bipartite entanglement

Class problems

Resource Theories.

1. a) Argue that any resource theory defines a partial order $\rho \preceq \sigma$ on the set of all quantum states $\mathcal{D}$ based on whether or not we can send $\rho \to \sigma$ under some free operation $\mathcal{E} \in F$.

   Answer 1.a) A partial order on a set is an arrangement such that, for certain pairs of elements, one precedes the other. A resource theory is defined by its set of free operations. For any pair of states A and B it is either possible to freely transform A to B or B to A or not possible to transform in either direction. Hence a resource theory defines a partial order.

   b) How can the notion of a resource measure M be defined using this partial ordering?

   Answer 1.b) A resource measure is property of a quantum state (i.e. a map from quantum states to a real number) which is non-increasing under free operations. Hence this partial ordering can be used as a necessary condition in order to identify potential resource measures.

   c) Show that any two free states σ_1 and σ_2 are equal under the partial ordering.

   Answer 1.c) For free states σ_i $(i = 1, 2)$, we have $1 \leftrightarrow \sigma_i$. By transitivity, we have $\sigma_1 \leftrightarrow \sigma_2$ (i.e. we can create one state to the other under some free operation). Using the property of the resource measure $M(\sigma) \geqslant M(\mathcal{E}(\sigma))$ for any quantum state σ and any free operations $\mathcal{E}$, we have $M(\sigma_1) = M(\sigma_2)$ for any two states σ_1 and σ_2 that are equal under partial ordering i.e. if $\sigma_1 \leftrightarrow \sigma_2$ (see next question).

   d) Show that any resource measure M must ascribe the same value to any two states that are equal under this ordering. (Does the converse hold?)

   Answer 1.d) This is what we used before. Indeed, if two states σ_1 and σ_2 are equal under partial ordering, we have two free operations $\mathcal{E}_1, \mathcal{E}_2$ such that $\sigma_2 = \mathcal{E}_1(\sigma_1)$ and $\sigma_1 = \mathcal{E}_2(\sigma_2)$, thus $M(\sigma_1) = M(\mathcal{E}_2(\sigma_2)) \leqslant M(\sigma_2) = M(\mathcal{E}_1(\sigma_1) \leqslant M(\sigma_1)$ which implies $M(\sigma_1) = M(\sigma_2)$. The converse is not always true !

2. Make up your own resource theory! (Optional)

   Answer 2: Examples https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.116.120404, https://arxiv.org/abs/1804.10190 and (note the publication date) https://arxiv.org/abs/1903.12629.

3. Prove that it is possible for Bob to perform a local operation conditional on an outcome of Alice's measurement via LOCC. That is, show it is possible to perform $\rho_{AB} \to \sum_i (M_i \otimes U_i)\rho_{AB}(M_i^\dagger \otimes U_i^\dagger)$ via LOCC.

   Answer 3: Alice does some measurement $\{M_i\}$ and gets outcome i, she then tells Bob the value of i and conditional on this, Bob apply unitary U_i on his half of the state. See lecture notes for more mathematical detail.

4. Prove that product states are the states that can be prepared freely via LOCC.

   Answer 4: For example, we can use previous exercise with input state $\rho_{AB} = |0_A, 0_B\rangle\langle 0_A, 0_B|$ and easily show that we get product state. To be more general, instead of applying unitaries Bob may apply any quantum operations on his half of the state i.e. same as previous exercise but Bob would perform operations of the form $\mathbb{1}_A \otimes \mathcal{E}_B^{(i)}$ where $\mathcal{E}_B^{(i)}$ are quantum operation applied on B. You may easily get an output state of the form $\sum_i p_i \sigma_A^{(i)} \otimes \rho_B^{(i)}$.

Majorization.

1. Draw the partial order defined by majorization on the following vectors $\boldsymbol{v}_1 = (0, 1/3, 2/3)$, $\boldsymbol{v}_2 = (1/3, 1/3, 1/3)$, $\boldsymbol{v}_3 = (1/4, 1/5, 1/3)$, $\boldsymbol{v}_4 = (1/5, 3/5, 1/5)$, $\boldsymbol{v}_5 = (2/5, 1/2, 1/10)$, and $\boldsymbol{v}_6 = (0, 1, 0)$.

   Answer 1: We can draw a diagram with $v_6 \preceq v_1 \preceq v_4 \preceq v_2$ and $v_6 \preceq v_1 \preceq v_5 \preceq v_2$ (be careful, there is no majorization relation between v_4 and v_5). And v_3 does not sum to 1, so no relation with the rest.

2. Show that $(1/N, 1/N, ..., 1/N) \preceq \boldsymbol{p} \preceq (1, 0, ..., 0)$ for any probability distribution $\boldsymbol{p}$.

   Answer 2: Let us define $S_k = \sum_{i=1}^{k} x_i$ and $T_k = \sum_{i=k+1}^{N} x_i$ (with $S_k + T_k = 1$ and $x_1 \geqslant x_2 \geqslant ... \geqslant x_N$). So, by construction we have $S_k \geqslant k x_k$ and $T_k \leqslant (N-k) x_k$ for $k = 1, 2, ..N-1$ which implies $T_k \leqslant \frac{N-k}{k} S_k$. Finally, this leads to $1 = T_k + S_k \leqslant \frac{N}{k} S_k$ i.e. $S_k \geqslant k/N$ i.e. any distribution majorizes the uniform distribution. Showing that $(1, 0, ..., 0)$ majorizes every distribution is trivial ($S_k \leqslant 1$).

3. A useful equivalent definition of majorization is that $\boldsymbol{x} \preceq \boldsymbol{y}$ iff $\boldsymbol{x}$ is a convex combination of vectors obtained by permuting coordinates of $\boldsymbol{y}$.

   Use this fact to show that the diagonal elements of a density operator are majorized by its eigenvalues. (Super useful property!)

   (Hint use the fact that any doubly stochastic matrix can be written as a convex combination of permutation matrices).

   Answer 3: Consider the state $\rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ and let us define the doubly stochastic matrix P with coefficients $P_{ji} = |\langle j|\lambda_i\rangle|^2$. Thus, the diagonal of ρ is given by $\boldsymbol{s} = P\boldsymbol{\lambda}$ (where $\boldsymbol{\lambda}$ is the vector containing eigenvalues of ρ).Now, using the equivalent definition of majorization given above (and the hint), you can easily show that $\boldsymbol{s} \preceq \boldsymbol{\lambda}$.

Bipartite Entanglement.

1. Argue that $|\psi_-\rangle_{AB}$ can be transformed into any state $|\phi\rangle_{AB}$ via LOCC using Nielson's Majorization Theorem. Describe a protocol to do this in practise.

   Answer 1: The reduce state of Bell states is the maximally mixed state and its spectrum correspond to the uniform distribution. We have shown before that every distribution majorizes the uniform distribution, thus Bell states can be transformed into any states by Nielson's theorem. The protocol is described in the lecture (just need to add local rotations).

2. Show that transforming between the states $|\phi\rangle = \sqrt{\frac{15}{100}}|00\rangle + \sqrt{\frac{3}{10}}|11\rangle + \sqrt{\frac{4}{10}}|22\rangle + \sqrt{\frac{15}{100}}|33\rangle$ and $|\psi\rangle = \sqrt{\frac{3}{10}}|00\rangle + \sqrt{\frac{3}{10}}|11\rangle + \sqrt{\frac{3}{10}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle$ is not possible deterministically via LOCC.

   Answer 2: The (ordered) spectrum of the reduced states are respectively $\boldsymbol{\phi} = \frac{1}{100}(40, 30, 15, 15)$ and $\boldsymbol{\psi} = \frac{1}{100}(30, 30, 30, 15)$, then you can easily show (as follows) that there is no majorization relation between them. Again defining $S_k(\boldsymbol{x}) = \sum_{i=1}^{k} x_i$, here we have $S_k(\boldsymbol{\psi}) < S_k(\boldsymbol{\phi})$ for $k = 1, 2$ but $S_3(\boldsymbol{\psi}) > S_3(\boldsymbol{\phi})$. Thus transformations between them are not possible deterministically via LOCC from Nielson's Theorem.