Solutions Problem Sheet 3

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. Purifications

1. Compute purifications for the following states:

   a) $\rho_1 = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$

   b) $\rho_2 = 1/2(|0\rangle\langle 0| + |+\rangle\langle +|)$

   c) $\rho_3 = 1/2(|\psi_+\rangle\langle\psi_+| + |\phi_-\rangle\langle\phi_-|)$

   Answer 1: The purified states in system S are obtained using an ancilla subsystem A (single qubit here). The purified states are respectively:

   a) $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S \otimes |0\rangle_A + |1\rangle_S \otimes |1\rangle_A)$

   b) $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle_S \otimes |0\rangle_A + |+\rangle_S \otimes |1\rangle_A)$

   c) $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle_S \otimes |0\rangle_A + |\phi_-\rangle_S \otimes |1\rangle_A)$
   where tracing out subsystem A leads to the desired mixed states.

2. Consider the single qubit state $\rho = \frac{1}{2}(\mathbb{I} + 0.1X + 0.1Y + 0.2Z)$
   a) Write ρ as a matrix in the computational basis.
   b) Compute the eigen-decomposition of ρ.
   c) Is ρ mixed or pure? How do you know?
   d) Compute a pure state decomposition of ρ involving three states.
   e) Hence state i. a purification of ρ using a single qubit environment and ii. a purification using a qutrit environment.

   Answer 2: a) In the computational basis $\rho = \frac{1}{20}\begin{pmatrix} 12 & 1-i \\ 1+i & 8 \end{pmatrix}$.

   b) The eigenvalues of ρ are $\lambda_\pm = \frac{1}{2} \pm \frac{1}{10}\sqrt{\frac{3}{2}}$. They are respectively associated to eigenvectors $|\psi_+\rangle = \frac{1}{\sqrt{3+\sqrt{6}}}\left((1+\sqrt{\frac{3}{2}})|0\rangle + \frac{(1+i)}{2}|1\rangle\right)$ and $|\psi_-\rangle = \frac{1}{\sqrt{3-\sqrt{6}}}\left((1-\sqrt{\frac{3}{2}})|0\rangle + \frac{(1+i)}{2}|1\rangle\right)$.

   c) The eigenvalues of a pure state are 1 and 0 (in dimension d, 0 is degenerated $d-1$ times). In this case, the eigenvalues are not 0 and 1. Equivalently, we can argue by showing that the purity is $\mathrm{Tr}[\rho^2] = 0.53 < 1$.

   d+e) Here we first consider the purification of the state acting on S on space using a qubit environment A. Such as $|\psi\rangle = \sqrt{\lambda_+}|\psi_+\rangle_S \otimes |0\rangle_A + \sqrt{\lambda_-}|\psi_-\rangle_S \otimes |1\rangle_A$. Then, we can extend A to a qutrit environment with basis $\{|0\rangle_A, |1\rangle_A, |2\rangle_A\}$. Applying any unitary U_A acting on the subsystem A (i.e. $\mathbb{1}_S \otimes U_A$) will result in the same mixed state after partial trace over A. For example, we can use the unitary of the Fourier transform given by $U_A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$ where $w = e^{2\pi i/3}$. Thus, an example of the purification of ρ using 3 states is given by

$|\tilde{\psi}\rangle = \sum_{i=0}^{2} \frac{1}{\sqrt{3}} \underbrace{\left(\sqrt{\lambda_+}|\psi_+\rangle_S + w^i \sqrt{\lambda_-}|\psi_+\rangle_S\right)}_{|\tilde{\psi}_i\rangle} \otimes |i\rangle_A$, where the $|\tilde{\psi}_i\rangle$ with $i = 0, 1, 2$ are these 3 pure states.

2. Measurements

3. Explain what is the difference between POVM measurements, projective measurements and a measurement of an observable.

Answer 3: A POVM is a set of positive semi-definite operators $\{M_i\}$ that sum to identity i.e. $\sum_i M_i = \mathbb{1}$. The POVM corresponds to a question you can ask of the quantum system and the individual operators the different answers you can obtain. The probability of obtaining answer i for state ρ is given by $\mathrm{Tr}[M_i \rho]$.

A projective measurement is a special type of POVM where the operators $\{\Pi_i\}$ form an orthonormal set, i.e. $\Pi_i \Pi_j = \delta_{i,j} \Pi_i$, and thus project onto certain basis.

In the case where we can associate the answer i to a projective measurement with a numerical value λ_i we can define an *observable*, i.e. a Hermitian matrix, corresponding to the measurement outcome $O = \sum_i \lambda_i \Pi_i$. This observable can then be used to directly compute properties of any state ρ corresponding to the quantity associated with the observable, e.g. one can compute the k_{th} moments of the quantity via $\langle O^k \rangle = \mathrm{Tr}[O^k \rho]$.

4. a) Write down a POVM measurement $\mathcal{M}$ that asks the question: "Is the system in the $|\Phi_+\rangle$ Bell state?"
b) Consider the 2-qubit separable state $|\psi\rangle \otimes |\psi\rangle$. What is the probability to find the system in $|\Phi_+\rangle$?

Answer 4:
(a) We have $\mathcal{M} = \{M_1 = |\phi_+\rangle\langle\phi_+|, M_2 = \mathbb{1} \otimes \mathbb{1} - |\phi_+\rangle\langle\phi_+|\}$, then the answer is "yes" if we measure M_1 and the answer is "no" if we measure M_2. Notice that $M_2 = |\psi_-\rangle\langle\psi_-| + |\phi_-\rangle\langle\phi_-| + |\psi_+\rangle\langle\psi_+|$.

(b) It is given by $|\langle\phi_+|(|\psi\rangle \otimes |\psi\rangle)|^2 = \frac{1}{2}\left(|\langle 0|\psi\rangle|^4 + |\langle 1|\psi\rangle|^4 + 2\mathrm{Re}[\langle 0|\psi\rangle^2 \langle\psi|1\rangle^2]\right)$.

5. Consider a d dimensional system S and $\mathcal{M} = \{\alpha_0 \mathbb{I}, \alpha_1 \mathbb{I}, \alpha_2 \mathbb{I}, \alpha_3 \mathbb{I}\}$. Is this a valid measurement? If so, what does the measurement do?

Answer 5: It is a valid measurement only if $\forall i$ we have real non-negative α_i and $\sum_{i=0}^{3} \alpha_i = 1$. One measures i with probability α_i for any state in S. (This can be viewed as a box with four different lights on top. For any state you feed in the i_{th} light turns on with probability α_i).

6. Suppose Bob hands you a quantum state and promises that it is either $|\phi_1\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ or $|\phi_2\rangle = \cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$.
a) Sketch these states on the Bloch sphere.
b) Design a measurement that perfectly distinguishes the states *some* of the time, is inconclusive at others, but never makes a mistake.
c) What is the probability in a *single* run of the experiment that you guess correctly?
d) Write down a projective measurement on a larger system that can be used to realise this POVM.

Answers 6:
(a) In term of the Bloch vector i.e. coordinates (x,y,z) on the Bloch sphere these states are given by $(\sin(\theta), 0, \cos(\theta))$ and $(-\sin(\theta), 0, \cos(\theta))$ respectively for $|\phi_1\rangle$ and $|\phi_2\rangle$.

(b) These 2-states are not orthogonal, but we can design a measurement that contains the 2 projectors on their respective orthogonal states that are their reflection w.r.t the Bloch sphere centre i.e. $|\phi_1^\perp\rangle = \sin(\theta/2)|0\rangle - \cos(\theta/2)|1\rangle$ and $|\phi_2^\perp\rangle = \sin(\theta/2)|0\rangle + \cos(\theta/2)|1\rangle$. Then, we can chose
$\mathcal{M} = \{M_0 = \frac{|\cos(\theta)|+\cos(\theta)}{1+|\cos(\theta)|}|0\rangle\langle 0| + \frac{|\cos(\theta)|-\cos(\theta)}{1+|\cos(\theta)|}|1\rangle\langle 1|, M_1 = \frac{1}{1+|\cos(\theta)|}|\phi_1^\perp\rangle\langle\phi_1^\perp|, M_2 = \frac{1}{1+|\cos(\theta)|}|\phi_2^\perp\rangle\langle\phi_2^\perp|\}$. So if the outcome is 0 we don't know what is the state, but if the outcome is 1 then we know that the state is $|\phi_2\rangle$ and inversely if the outcome is 2 then we know that the state is $|\phi_1\rangle$. Notice that this choice of the prefactors is

obtained by assuming that both projectors onto $|\phi_1^\perp\rangle$ and $|\phi_2^\perp\rangle$ shared the same coefficient (same probabilities). Then, we maximise this probability with the constraint that the three operators sums to identity and are positive semi-definite (POVM properties).

(c) In a single run, the probability of guessing correctly is $1-\cos(\theta)$ which is the probability of not having the outcome 0 (i.e. the probability of having either 1 or 2). Notice that this is the same in both cases ($|\phi_1\rangle$ and $|\phi_2\rangle$) and we also see that if $\theta=0$, then this probability is indeed 0 as $|\phi_1\rangle=|\phi_2\rangle$.

(d) We can also design this POVM measurement using projectors on a 2-qubits system (using an ancilla qubit) such as $\mathcal{M}=\{|0\rangle\langle 0|\otimes|\phi_1^\perp\rangle\langle\phi_1^\perp|, |1\rangle\langle 1|\otimes|\phi_2^\perp\rangle\langle\phi_2^\perp|, |0\rangle\langle 0|\otimes|\phi_1\rangle\langle\phi_1|, |1\rangle\langle 1|\otimes|\phi_2\rangle\langle\phi_2|\}$. (We can easily check that these 4 operators projects onto 4 orthogonal vectors that forms a basis of the 2-qubit Hilbert space and so they sum to identity i.e. $\mathbb{1}\otimes\mathbb{1}$).

7. Propose an 'informationally complete' measurement for a single qubit state. That is, a POVM measurement $\mathcal{M}$ that allows you to perfectly reconstruct a single qubit quantum state. What about a 2-qubit state?

Answer 7: To perfectly reconstruct a single qubit state, we need to know its Bloch vector. So, we can perform a convex mixture of projectors onto the eigen-basis of the 3 Paulis X, Y and Z :

$$\mathcal{M}=\left\{\frac{1}{3}|0\rangle\langle 0|, \frac{1}{3}|1\rangle\langle 1|, \frac{1}{3}|+\rangle\langle +|, \frac{1}{3}|-\rangle\langle -|, \frac{1}{3}|y+\rangle\langle y+|, \frac{1}{3}|y-\rangle\langle y-|\right\}$$

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Notice that Z and the identity share the same basis. For the 2-qubit state, we have all possible basis elements for the Paulis on 2-qubits which is equal to $6^2=36$ (6^n for n qubits). Indeed, $\{X,Y,Z\}^{\otimes 2}$ contains 9 elements and each element has 4 distinct eigenvectors (the 9 Paulis have no common eigenvectors) so 36 in total. Here the coefficient multiplying each projector is 1/9 (we have 9 distinct basis to measure and each of the 4 projectors per basis sums to identity).

8. Suppose $\mathcal{M}=\{M_i\}_{i=1}^m$ and $\mathcal{N}=\{N_i\}_{i=1}^n$ are two different POVM measurements.
a) We can define an $m+n$ outcome POVM from $\mathcal{M}$ and $\mathcal{N}$ by flipping a biased coin and with probability p doing $\mathcal{M}$ and probability $(1-p)$ doing $\mathcal{N}$. Write down the measurement operators for this measurement.
b) Suppose now $m=n$. We can alternatively define a m measurement composed of the operators $\{pM_i+(1-p)N_i\}_{i=1}^m$. How does this measurement differ from the one in part (a)? How could you realise it?

Answer 8:

(a) Here the measurement operators are $\mathcal{X}=\{pM_i|i=1,2,...,m\}\bigcup\{(1-p)N_i|i=1,2,...,n\}$.

(b) Here, instead of having $2m$ distinct operators as in (a), we only have m that are obtained by summing (with weight) operators in $\mathcal{M}$ and $\mathcal{N}$. We have less information here as we cannot distinguish measurements M_i and N_i after obtaining outcome i. We can imagine something similar than a), but with no information whether we measured M_i or N_i, instead we only know the outcome i.