Solutions Problem Sheet 4: Quantum Channels (Part 1) and Naimark's Theorem

Class problems

1. Consider the matrices:

$$A_0 \propto \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A_1 \propto \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{1}$$

a) Show that A_0 and A_1 are Kraus operators of a quantum operation given the right normalization factor.

b) What are the dimensions of the input quantum system and the output quantum system?

Answer:

a) We fulfill Kraus condition i.e. $E_0^\dagger E_0 + E_1^\dagger E_1 = \mathbb{1}$ iif $E_i = \frac{1}{\sqrt{6}} A_i$ for $i = 0, 1$.

b) The input is dimension 2 and the output is dimension 4.

2. Suppose we have a qubit system A interacting with a qubit environment through the unitary

$$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes X_B \tag{2}$$

a) Write down the channel induced on the system assuming that the environment qubit starts in the state $|0\rangle$.

Suppose now

$$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes e^{-i\lambda t X_B} \ ? \tag{3}$$

b) Write down the channel induced on the system in this case.

c) Comment on what this tells you about the link between system and environment interactions and decoherence.

Answer:

a) The environment qubit starts in the state $|0\rangle$, so the channel is given by $\mathcal{E}(\rho_A) = \mathrm{Tr}_B[U(\rho_A \otimes |0\rangle\langle 0|_B)U^\dagger]$. Thus we have $\mathcal{E}(\rho_A) = \langle 0|\rho_A|0\rangle |0\rangle\langle 0|_A + \langle 1|\rho_A|1\rangle |1\rangle\langle 1|_A$. This channel corresponds to Kraus operators $\{|0\rangle\langle 0|_A, |1\rangle\langle 1|_A\}$.

b) Following the same reasoning, we see that the state after the channel is such that the diagonal is unchanged but in this case the off-diagonal elements are multiplied by $\cos(\lambda t)$. For example, we can choose Kraus operators $\{\sqrt{\cos(\lambda t)}\mathbb{1}, \sqrt{1-\cos(\lambda t)}|0\rangle\langle 0|_A, \sqrt{1-\cos(\lambda t)}|1\rangle\langle 1|_A\}$.

We can also notice that it corresponds to the dephasing channel with Kraus operators $\{\sqrt{\frac{1+\cos(\lambda t)}{2}}\mathbb{1}, \sqrt{\frac{1-\cos(\lambda t)}{2}}Z\}$ (you can check that it indeed gives the same as a) with $\lambda t = \pi/2$).

c) This example shows well (in a mathematical way) how interactions with the environment induces decoherence. As said before, it is a dephasing channel and it is well known that it induces decoherence. The one obtained in a) is the completely dephasing channel that induce complete loss of coherence. In b), we have a dependence on parameters λt that tunes the decoherence. (Be careful about the fact that the channel output is defined for a fixed t and λ and do not interpret it as a time evolution. Indeed applying the channel to a fully decohered state has no effect. If we want to apply the channel recursively, we need to change the input at every step.)

3. Suppose we have a qubit system A interacting with a qubit environment through the unitary

$$U = \frac{1}{\sqrt{2}} \left(X_A \otimes \mathbb{I}_B + Y_A \otimes X_B \right) . \tag{4}$$

a) What is the channel induced on the system in this case assuming that the environment starts in the state $|0\rangle$.

b) By looking at the action of the channel on a Bloch vector describe the action of this channel.

Answer:

a) The channel is given by $\mathcal{E}(\rho_A) = \frac{1}{2} X_A \rho_A X_A + \frac{1}{2} Y_A \rho_A Y_A$ where the Kraus operators are $\{\frac{1}{\sqrt{2}} X_A, \frac{1}{\sqrt{2}} Y_A\}$.

b) The action on the Bloch vector is such that $(x, y, z) \to (0, 0, -z)$. So, it's a projection of the state on the z-axis and a reflection w.r.t the plane $z = 0$ ($z \to -z$). You can also see it by noticing that the of diagonal terms of the density matrix vanishes and the diagonal terms are swapped.

4. *Following Question 6 from the previous problem sheet.*

Suppose Bob hands you a quantum state and promises that it is either $|\phi_1\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ or $\cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$.

A measurement that can distinguish these states perfectly some of the time (and is inconclusive at other times) is given by the following measurement operators:

$$\mathcal{M} = \{M_0 = \frac{|\cos(\theta)| + \cos(\theta)}{1 + |\cos(\theta)|} |0\rangle\langle 0| + \frac{|\cos(\theta)| - \cos(\theta)}{1 + |\cos(\theta)|} |1\rangle\langle 1|, M_1 = \frac{1}{1 + |\cos(\theta)|} |\phi_1^\perp\rangle\langle\phi_1^\perp|, M_2 = \frac{1}{1 + |\cos(\theta)|} |\phi_2^\perp\rangle\langle\phi_2^\perp|\}$$

d) Write down a projective measurement on a larger system that can be used to realise this POVM.

Answer: (d) We can also design this POVM measurement using projectors on a 2-qubits system (using an ancilla qubit) such as $\mathcal{M} = \{|0\rangle\langle 0| \otimes |\phi_1^\perp\rangle\langle\phi_1^\perp|, |1\rangle\langle 1| \otimes |\phi_2^\perp\rangle\langle\phi_2^\perp|, |0\rangle\langle 0| \otimes |\phi_1\rangle\langle\phi_1|, |1\rangle\langle 1| \otimes |\phi_2\rangle\langle\phi_2|\}$. (We can easily check that these 4 operators projects onto 4 orthogonal vectors that forms a basis of the 2-qubit Hilbert space and so they sum to identity i.e. $\mathbb{1} \otimes \mathbb{1}$).

5. Take a look at *Ancilla-free implementation of generalized measurements for qubits embedded in a qudit space* (Phys. Rev. Research 4, 033027 (2022)).

a) Why and how does this paper use 'informationally complete' POVMs?

b) How is Naimark's theorem used to realize these measurements?

c) Discuss this paper (no need to write anything down). Do you think it's any good? Any ideas about how you might like to extend it?

d) Got time to kill? Have a go at implementing their proposal!

Answer: Here the informationally complete (IC) POVM is defined as a POVM that allows to estimate any hermitian observables (by estimating the measurement operators probabilities). From Naimarks theorem, there exists a way of performing the POVM (IC-POVM here) with a projective measurement on an extended Hilbert space (either by a tensor product extension (TPE) or by direct sum extension (DSE) as explain in the text). The TPE may double the total number of qubit by adding ancilla register. Instead of this, they proposes to use qudit representation instead of qubit that allows then to perform a DSE and thus avoid unused degree of freedom. They proposed an experimental implementation of the Stinespring dilation unitary using laser pulse that induces transition between adjacent levels. This is a really quick summary of the paper and there is much more to say about it, but it gives you an example of the usefulness of Naimark's theorem in practice. Indeed, in general (if not always) we perform projective measurement in practice as with quantum circuits.