Solutions Problem Sheet 13: Entanglement Theory (Part 2)

Class problems

1. *Asymptotic entanglement distillation.* Roughly how many singlets are needed to construct 100 copies of the state $|\phi\rangle = \alpha(2|01\rangle - 3|+0\rangle + 5|22\rangle)$, where α is a normalization constant (that you have to determine) and $|+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$, under LOCC?

   Answer 1: The spectrum of the reduced state is approximately $\boldsymbol{\phi} \approx (0.035, 0.205, 0.760)$, thus the entropy of the reduced state is $S \approx 0.9396$. Therefore, we need 94 singlets to create $N = 100$ copy of the state (as $NS \approx 93.96$).

2. *Hyperplanes.* In $\mathbb{R}^3$ determine the hyperplanes defined by the vector $\boldsymbol{v}$ for the cases a) $\boldsymbol{v} = (0, 1, 0)$, b) $\boldsymbol{v} = (0, -2, 0)$ and c) $\boldsymbol{v} = (1, 1, -1)$. In each case determine the 'positive' and 'negative' sides of the plane.

   Answer 2: Straightforward with the definition of the hyperplane. The points $\boldsymbol{r} = (x, y, z)$ in the hyper plane given by $\boldsymbol{v}$ satisfy $\boldsymbol{v} \cdot \boldsymbol{r} = xv_1 + yv_2 + zv_3 = 0$ and the positive side is all points s.t. $\boldsymbol{v} \cdot \boldsymbol{r} > 0$ (< 0 for the negative sign). a) XZ-plane with positive with $y > 0$, negative $y < 0$ b) same plane as a) with inverse relation for positive/negative sides. c) plane $x + y - z = 0$ with positive (negative) side in volume $x + y - z > 0$ (< 0).

3. *Superoperators to detect entanglement.*

   Define the maximally entangled state $|\Omega\rangle = |\text{vec}(\mathbb{I})\rangle$ and the Choi state associated with an operator $\mathcal{E}$ as $J(\mathcal{E}) = \mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)$.

   a) Show that $|\Omega\rangle\langle\Omega|^{T_A} = \text{SWAP}$.
   Answer 3.a) Straightforward $J(\mathcal{E}) = \sum_{i,j} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$ with $\mathcal{E}(.) = (.)^T$.

   b) Show that $\text{Tr}[A \otimes B\,\text{SWAP}] = \text{Tr}[AB]$.
   Answer 3.b) Also straightforward: $\text{Tr}[A \otimes B\,\text{SWAP}] = A_{ij}B_{kl}\text{Tr}[|i\rangle\langle l| \otimes |k\rangle\langle j|] = A_{ij}B_{ji} = \text{Tr}[AB]$ (implicit summation over indices).

   c) Assuming that every positive superoperator $\mathcal{E}$ has a decomposition $\mathcal{E}(X) = \sum_i A_i X B_i^\dagger$ deduce that we can define a 'dual map' $\mathcal{E}^*$ such that $\text{Tr}[\mathcal{E}(\rho)B] = \text{Tr}[\rho\mathcal{E}^*(B)]$ for all states ρ and Hermitian operators B.
   Answer 3.c) Using linearity and cyclicity respectively leads to $\text{Tr}[\mathcal{E}(\rho)B] = \sum_i \text{Tr}[A_i\rho B_i^\dagger B] = \sum_i \text{Tr}[\rho B_i^\dagger B A_i] = \text{Tr}[\rho\mathcal{E}^*(B)]$ with $\mathcal{E}^*(X) = \sum_i B_i^\dagger X A_i$.

   d) Prove that

   $$\text{Tr}[(B \otimes \sigma^T)J(\mathcal{E})] = \text{Tr}[\mathcal{E}(\sigma)B] \tag{1}$$

   Answer 3.d) We have $\text{Tr}[(B \otimes \sigma^T)J(\mathcal{E})] = \text{Tr}[(B \otimes \sigma^T)\mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)] = \text{Tr}[(\mathcal{E}^*(B) \otimes \sigma^T)|\Omega\rangle\langle\Omega|)] = \text{Tr}[(\mathcal{E}^*(B) \otimes \sigma)|\Omega\rangle\langle\Omega|^{T_B}] = \text{Tr}[(\mathcal{E}^*(B) \otimes \sigma)SWAP] = \text{Tr}[\mathcal{E}^*(B)\sigma] = \text{Tr}[\mathcal{E}(\sigma)B]$ where the first and the second equality can be easily shown, the third one is obtained by applying partial transpose (on the second register called B here) and then we use a), b) and c) respectively. Notice that from the second equality we could also use the relations $|vec(AXB)\rangle = A \otimes B^T|vec(X)\rangle$ (ricochet property with $X = \mathbb{1}$) and $\text{Tr}[O \otimes \mathbb{1}|\Omega\rangle\langle\Omega|] = \text{Tr}[O]$ to show $\text{Tr}[(\mathcal{E}^*(B) \otimes \sigma^T)|\Omega\rangle\langle\Omega|)] = \text{Tr}[(\mathcal{E}^*(B)\sigma \otimes \mathbb{1})|\Omega\rangle\langle\Omega|] = \text{Tr}[\mathcal{E}^*(B)\sigma]$.

   Let use define the super-operator $\Phi_{\mathcal{J}}(\rho) = \text{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{J}]$

   e) Show that $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$ and $\mathcal{J}(\Phi_{\mathcal{J}(\mathcal{E})}) = \mathcal{J}(\mathcal{E})$. How does this connect to Eq. (1)?

   Answer e) This is very similar to what we have done before. Notice that $\text{Tr}_2[O \otimes \mathbb{1}|\Omega\rangle\langle\Omega|] = O$, thus by using first the ricochet property and this property we have $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \text{Tr}_2[\mathcal{E}(\rho) \otimes \mathbb{1}] = \mathcal{E}(\rho)$, so we have shown that

$\Phi_{\mathcal{J}(\mathcal{E})} = \mathcal{E}$ which also proves $\mathcal{J}(\Phi_{\mathcal{J}(\mathcal{E})}) = \mathcal{J}(\mathcal{E})$. We can directly obtain Eq. (1) from these relations. You can also use more brute force method as follows: $\mathrm{Tr}_2[(\mathbb{1} \otimes \rho^T)\mathcal{J}(\mathcal{E})] = \sum_{i,j} \mathrm{Tr}_2[(\mathbb{1} \otimes \rho^T)(\mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|)] = \sum_{i,j} \mathcal{E}(|i\rangle\langle j|)\langle j|\rho^T|i\rangle = \sum_{i,j} \mathcal{E}(|i\rangle\langle j|)\langle i|\rho|j\rangle = \mathcal{E}(\rho)$ (notice that $\mathrm{Tr}_2[A \otimes B] = A\mathrm{Tr}[B]$).

f) What is the connection between SWAP and the Peres-Horodecki criterion?

Answer f): The SWAP operator is the entanglement witness corresponding to the Peres-Horodecki criterion.

That is, $\mathrm{Tr}[\rho\,\mathrm{SWAP}] < 0$ implies ρ^{T_B} has a negative eigenvalue. To see this note that $\mathrm{Tr}[\rho\mathrm{SWAP}] = \mathrm{Tr}[\rho^{T_B}|\Omega\rangle\langle\Omega|]$. Writing ρ^{T_B} in terms of its eigendecomposition it follows that if $\mathrm{Tr}[\rho\,\mathrm{SWAP}] < 0$ then ρ^{T_B} has a negative eigenvalue.