Solutions Problem Sheet 2: Quantum Basics

Don't initially use mathematica for this! (But maybe use it to double check your answers).

Quantum Theory

1. The state space of a single qubit can be represented by a Bloch vector $\boldsymbol{r}$ with norm less than 1, i.e. $|\boldsymbol{r}| \leqslant 1$.

   A pure state is a state that cannot be written in the form $\rho = p\sigma_0 + (1-p)\sigma_1$ for any two states σ_0, σ_1 and any $0 < p < 1$.

   a) Argue (geometrically!) that any pure state has a Bloch vector of norm 1 and hence can be written as $|\psi\rangle\langle\psi|$.

   b) Sketch on the Bloch sphere the states: $|1\rangle$, $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|+y\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, and $\frac{1}{\sqrt{5}}(|0\rangle + i\sqrt{4}|1\rangle)$.

   c) $\mathrm{Tr}[\rho^2]$ is known as the purity of a state. Argue why this is an appropriate name.

   Solution:
   a) As seen in class, the state $\rho = p\sigma_0 + (1-p)\sigma_1$ with $0 < p < 1$ is necessarily between σ_0 and σ_1 which means that it is necessarily a point inside the sphere i.e. with $r < 1$. In other word, we can only have $r = 1$ when $\sigma_0 = \sigma_1$ or $p = 0$ or $p = 1$ which just means that we cannot write ρ as $\rho = p\sigma_0 + (1-p)\sigma_1$ with $0 < p < 1$ for any two (different) states σ_0 and σ_1.

   b) The associated coordinates (x, y, z) on the Bloch sphere (BS) are: $(0, 0, -1)$ for $|1\rangle$, $(1, 0, 0)$ for $|+\rangle$, $(0, 1, 0)$ for $|+y\rangle$, and $(0, 4/5, -3/5)$ for the state $\frac{1}{\sqrt{5}}(|0\rangle + i\sqrt{4}|1\rangle)$.

   c) We have $\mathrm{Tr}[\rho^2] = \frac{1+\boldsymbol{r}^2}{2}$. The centre of the BS, $\mathbb{1}/2$, is the maximally mixed state which corresponds to $r = 0$ and the pure states are on the sphere i.e. with $r = 1$. In other word, the closer we are to a pure state (mixed states) the larger (smaller) will be $\mathrm{Tr}[\rho^2]$.
   Notice that $\mathrm{Tr}[\rho^2]$ is a better quantity than r as it can be generalised to higher dimensions (not just 2). Indeed, for a state in dimensions d with eigenvalues $\{\lambda_i\}_{i=1}^d$, we have $\mathrm{Tr}[\rho^2] = \sum_{i=1}^d \lambda_i^2$ that is maximal when the λ_i are uniform (i.e. $1/d$) which corresponds to the maximally mixed state $\mathbb{1}/d$. And the maximum of the purity is reached for a pure state i.e. $\lambda_i = 0$ for a given i (and all other eigenvalues are 0 by definition).

2. Are the following states pure or mixed?

   a) $\rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$

   b) $\rho = \frac{1}{6}(3|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + 3|1\rangle\langle 1|)$

   Solution:
   A simple way to check whether a state is pure or mixed is to check if $\rho = \rho^2$.

   a) In this case ρ is pure (it is the state $|+\rangle$).

   b) In this case ρ is mixed ($\mathrm{Tr}[\rho^2] = 5/9$).

3. a) Sketch the points $\frac{1}{5}|0\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|$, $\frac{1}{3}|-y\rangle\langle -y| + \frac{2}{3}|0\rangle\langle 0|$ on the Bloch sphere.
   (A geometric answer will suffice I don't need the exact messy algebra).

   b) Give a geometric argument to show that $\frac{1}{2}\mathbb{I}/2$ can be written as a convex combination of an infinite number of pairs of states.

   c) Write $\frac{1}{2}\mathbb{I}$ as a convex combination of 4 states (easy). How about as a convex combination of three states?

   Solution:
   a) The point for state $\frac{1}{5}|0\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|$ corresponds to $\frac{1}{5}$ along z and $\frac{4}{5}$ along $-z$, i.e. the point is $(0, 0, -\frac{3}{5})$ inside of the Bloch sphere. The point for state $\frac{1}{3}|-y\rangle\langle -y| + \frac{2}{3}|0\rangle\langle 0|$ corresponds to $-1/3$ along y and $+2/3$ along z, i.e. the point $(0, -1/3, 2/3)$.

   b) The state $\frac{1}{2}\mathbb{I}$ is the centre of the Bloch sphere, so we can take any state and compute its superposition with its reflection w.r.t. to the centre of the Block sphere to obtain the identity. The simplest case is $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ (equivalently we can consider any basis for a single qubit state and add its reflection). We have to chose appropriate coefficients to ensure that the overall sums to $\frac{1}{2}\mathbb{1}$. So, each pairs of opposite states (projectors) are multiplied by a common coefficient and the sum of these coefficients should be 1/2. For example, with spherical coordinates (θ, ϕ) ($r = 1$ for pure states) states are $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ and its reflection corresponds to point point $(\phi+\pi, \pi-\theta)$ which indeed is its perpendicular state $|\psi^{\perp}\rangle \sin(\theta/2)|0\rangle - e^{i\phi}\cos(\theta/2)|1\rangle$ by averaging $\frac{1}{2}(|\psi\rangle\langle\psi| + |\psi^{\perp}\rangle\langle\psi^{\perp}|)$ over continuous variables θ and/or ϕ indeed leads to $\mathbb{1}/2$ and corresponds to an infinite sum of pairs states.

c) For example, we can write $\mathbb{1} = \frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|)$. For 3 states it is also much simpler to think geometrically. Any 3 points that forms an equilateral triangle on the sphere. For example, we can chose consider the pure states $|\psi_1\rangle = \cos(\pi/12)|0\rangle + i\sin(\pi/12)|1\rangle$ and $|\psi_2\rangle = \cos(5\pi/12)|0\rangle + i\sin(5\pi/12)|1\rangle$ such that $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = |y+\rangle\langle y+|$, so we have $\frac{1}{2}\mathbb{1} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |y-\rangle\langle y-|)$ (if you have another answer, trying to check where are these states on the BS to convince yourself can be a good exercise).

4. Given any quantum state ρ of a d-dimensional quantum system S, explain why we can write $\sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ where λ_k are real and positive and $\langle\lambda_k|\lambda_j\rangle = \delta_{jk}$. What does this tell us about how we can interpret ρ?

Solution:
This comes from the properties of density operators. As ρ is hermitian it directly follows from the spectral theorem that we can write $\rho = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ where λ_k are real and $\langle\lambda_k|\lambda_j\rangle = \delta_{jk}$. And as ρ is positive definite, λ_k are positive. Notice also that the eigenvalues sums to 1 because $\mathrm{Tr}[\rho] = 1$. It means that any (mixed) state ρ can be written as a statistical ensemble of pure states.

Class problems

1. Compute the partial trace ρ_A of the following states:

a) $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

b) $|\psi\rangle_{AB} \propto (\alpha|0+\rangle + \beta|11\rangle)$

c) $|\psi\rangle_{AB} \propto (|0+\rangle + 3|\psi 1\rangle + 5i|2-\rangle))$, where $|\psi\rangle$ is a state acting on A.

Solution:
We compute the partial trace over subsystem B to obtain the reduced state of subsystem A, formally

$$\begin{aligned}\rho_A =& \mathrm{Tr}_B\left(|\psi\rangle_{AB}\langle\psi|_{AB}\right) \\ =& \sum_{|i\rangle_B}(I_A \otimes \langle i|_B)|\psi\rangle_{AB}\langle\psi|_{AB}(I_A \otimes |i\rangle_B)\,,\end{aligned}$$

which gives:

a) $\rho_A = \mathbb{1}/2$.

b) $\rho_A = \frac{1}{|\alpha|^2+|\beta|^2}(|\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + \frac{\alpha\beta^*}{\sqrt{2}}|0\rangle\langle 1| + \frac{\alpha^*\beta}{\sqrt{2}}|1\rangle\langle 0|)$

c) $\rho_A = \frac{1}{35+\frac{3}{\sqrt{2}}(\langle 0|\psi\rangle+\langle\psi|0\rangle)+\frac{15i}{\sqrt{2}}(\langle 2|\psi\rangle-\langle\psi|2\rangle)}\Big(|0\rangle\langle 0| + 9|\psi\rangle\langle\psi| + 25|2\rangle\langle 2| + \frac{3}{\sqrt{2}}(|0\rangle\langle\psi| + |\psi\rangle\langle 0|) - \frac{15i}{\sqrt{2}}(|2\rangle\langle\psi| - |\psi\rangle\langle 2|)\Big)$

2. Optional! If you've done this before and are happy with it feel free to skip.

a) Use a series expansion to show that $e^{-i\theta\sigma_i/2} = \cos(\theta/2)\mathbb{1} - i\sin(\theta/2)\sigma_i$.

Solution:
As $\sigma_i^{2k+1} = \sigma_i$ and $\sigma_i^{2k} = \mathbb{1}$ for all integers k, then we use the serie expansion of the exponential, split it into

even and odd power terms and we recognise the sine and cosine series:

$$e^{-i\theta\sigma_i/2} = \sum_{k=0}^{\infty} \frac{(-i\theta/2)^k \sigma_i^k}{k!} = \mathbb{1} \sum_{n=0}^{\infty} \frac{(-i\theta/2)^{2n}}{(2n)!} + \sigma_i \sum_{k=0}^{\infty} \frac{(-i\theta/2)^{2n+1}}{(2n+1)!}.$$

Finally, we use $(-i)^{2n} = (-1)^n$ and $(-i)^{2n+1} = -i(-1)^n$ to get

$$e^{-i\theta\sigma_i/2} = \mathbb{1} \sum_{n=0}^{\infty} \frac{(-1)^n(\theta/2)^{2n}}{(2n)!} - i\sigma_i \sum_{k=0}^{\infty} \frac{(-1)^n(\theta/2)^{2n+1}}{(2n+1)!},$$

where we directly recognise expansions of $\cos(\theta/2)$ and $\sin(\theta/2)$.

b) What is the effect of evolving the state $|+\rangle$ under $e^{-i\theta\sigma_z/2}$ for $\theta = \pi/2$? State the final state and sketch this evolution on the Bloch sphere.

Solution:
The state $|+\rangle$ is just the unit vector on the x-axis (i.e. $(1,0,0)$) and $e^{-i\theta\sigma_z/2}$ for $\theta = \pi/2$ corresponds to a rotation of angle $\pi/2$ around the z axis, which brings the state to $|y+\rangle$ i.e. the vector $(0,1,0)$ on the BS.

c) What about evolving $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ under $e^{-i\theta\sigma_x/2}$ for $\theta = -3\pi/4$?
Solution:
Applying the rotation to states $|0\rangle$ and $|1\rangle$ leads to

$$e^{-i\theta\sigma_x/2}|0\rangle\langle 0|e^{+i\theta\sigma_x/2} = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}-1 & -i \\ i & \sqrt{2}+1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} - \frac{1}{\sqrt{2}}Z + \frac{1}{\sqrt{2}}Y)$$

and

$$e^{-i\theta\sigma_x/2}|1\rangle\langle 1|e^{+i\theta\sigma_x/2} = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}+1 & i \\ -i & \sqrt{2}-1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \frac{1}{\sqrt{2}}Z - \frac{1}{\sqrt{2}}Y).$$

So, the state ρ after the rotation becomes

$$e^{-i\theta\sigma_x/2}\rho e^{+i\theta\sigma_x/2} = \frac{1}{2}(\mathbb{1} + \frac{1-2p}{\sqrt{2}}Z - \frac{1-2p}{\sqrt{2}}Y).$$

Notice that applying rotation to Bloch vector $r_0 = (0,0,1)$ leads to $r_0' = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and as the initial state has Bloch vector $r = (0,0,2p-1)$ we just multiply r_0' by $2p-1$.