Solutions of Problem Sheet 1: Classical Theory

1. Let S be a single DNA base, which can be in one of four sharp states A,T,G,C.

   a) Define an appropriate set of basis states for this system.

   b) What is the state of the system if you know that the DNA strand is not C but otherwise are completely uncertain?

   c) Define a measurement to ask: "Is the base 'A or T'?"

   Solution:
   We use 4-dimensional probability vectors to describe the states of this DNA system.

   a) We define for those four sharp states A, T, G and C a set of probability vectors $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$, respectively. Then, any state of the system can be described by a vector $\boldsymbol{v}$ such that its components are probabilities of the corresponding basis states i.e.

   $$r = (\Pr(\text{“}A\text{”}), \Pr(\text{“}T\text{”}), \Pr(\text{“}G\text{”}), \Pr(\text{“}C\text{”}))\ .$$

   b) The state is $\frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3$.

   c) Here we design a measurement $\mathcal{M} = \{m_0 = (1, 1, 0, 0),\ m_1 = (0, 0, 1, 1)\}$. The measurement outcome 0 leads to answer "yes" and measuring 1 leads to answer "no" and we indeed have the outcome probabilities given by $p(i) = m_i \cdot r$ for $i = 0, 1$.

2. Consider the bars and stripes data set (shown below) as corresponding to different possible states of a classical system.

   a) Define an appropriate set of basis states for this system.

   b) Write down the classical states corresponding to the first three in the first line.

   c) What is the state space for this system?

   d) Define a measurement to determine whether the system is fully shaded.

   e) Define a measurement to determine whether the system contains blue or white stripes or bars, i.e. the first six patterns on the left.

   Solution:
   a) Here we have 16 different basis states. We can use a similar approach as before but in 16 dimensions such the basis states are given by $\{e_i\}_{i=0}^{15}$ (here e_i denotes the vector where the $(i+1)$-th component is 1 and all the others are 0, e.g. $e_0 = (1, 0, 0, ..., 0)$). For example, each element can be chosen such that i is the decimal value of the binary string corresponding to each image where a shaded (or unshaded) area corresponds to a digit of 1 (or 0, respectively), with the convention that we transfer the arrays line by line from top to bottom into bit-strings. Note that there are multiple conventions.

   b) The first three states in the first line are e_{15}, e_{10}, e_5 respectively. (From the top left to top right we have respectively $e_{15}, e_{10}, e_5, e_{11}, e_{14}, e_{13}, e_7, e_9$ and from the bottom left to the bottom right $e_0, e_{12}, e_3, e_4, e_1, e_2, e_8, e_6$.)

   c) It is a 16 dimensional state space where the components of probability vectors are real non-negative and their sum is 1, i.e. for a state $r = \sum_{i=0}^{15} r_i e_i$ we have $0 \leqslant r_i \leqslant 1$ for all i and $\sum_i r_i = 1$.

   d) If the system is fully shaded, the system is in state e_{15}. So, we can define a measurement

   $$\mathcal{M} = \{m_0 = e_{15},\ m_1 = \sum_{i=0}^{14} e_i\}\ ,$$

   such that we will obtain the answer "yes" for measuring 0 and "no" for measuring 1.

   e) Here, the states that contains visible stripes or bars are $e_0, e_3, e_5, e_{10}, e_{12}, e_{15}$. So, we define

   $$\mathcal{M} = \{m_0 = e_0 + e_3 + e_5 + e_{10} + e_{12} + e_{15},\ m_1 = e_1 + e_2 + e_4 + e_6 + e_7 + e_8 + e_9 + e_{11} + e_{13} + e_{14}\}$$

   such that we will obtain the answer "yes" for measuring 0 and "no" for measuring 1.

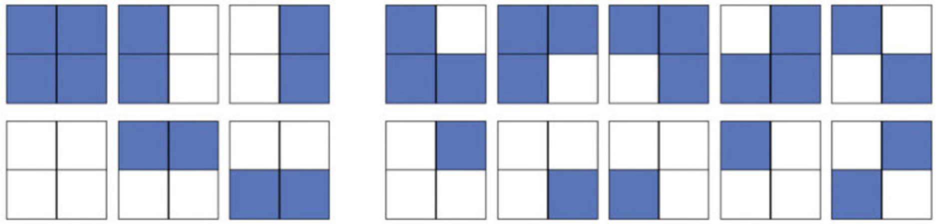

FIG. 1. **Bars and stripes data set.** Suppose each image here corresponds to a different state of a system.