Solutions Problem Sheet 9: Distinguishing quantum states and Matrix norms

Class problems

1. Suppose you are given a state ρ_0 with probability p and a state ρ_1 with probability $1-p$.

   a) What is the optimum binary POVM to answer the question 'Is the system in the state ρ_0 or ρ_1'?

   b) What is the probability of successfully distinguishing ρ_0 and ρ_1?

   Answers 1: We follow the same reasoning as proposed in the chapter 7 from lecture notes with $p = 1/2$, but here for an arbitrary $0 \leqslant p \leqslant 1$. We have the POVM operators $\{M_0, M_1\}$ which answer to the question (i.e. M_i if the state is ρ_i). We have $M_1 = \mathbb{1} - M_0$ and M_0 is decomposed into Pauli basis

$$M_0 = a(\mathbb{1} + bX + cY + dZ) \, ,$$

   for some real constants a, b, c, d where $a = 1/2$ and $\boldsymbol{x} \equiv (b, c, d)$ is in the Bloch sphere in order to have eigenvalues of M_i between 0 and 1 (recall previous exercises) i.e. M_i are two quantum states (and we will see that the probability of successfully guessing the state is maximised when M_i are pure states i.e. with Bloch vector of unit norm). The probability of successfully distinguishing these states is given by

$$\begin{aligned} p_{guess} &= \sum_{i=0,1} \mathrm{Pr}[\text{outcome i}|\, \rho_i] \cdot \mathrm{Pr}[\rho_i] && (1)\\ &= p\mathrm{Tr}[M_0\rho_0] + (1-p)\mathrm{Tr}[M_1\rho_1] && (2)\\ &= \frac{1}{2}\left(1 + \boldsymbol{x} \cdot (p\boldsymbol{x}_0 - (1-p)\boldsymbol{x}_1)\right) \, . && (3) \end{aligned}$$

   In the last equality we have replaced $M_1 = \mathbb{1} - M_0$ and used the Bloch representation for ρ_0 and ρ_1 defining their respective Bloch vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. We see that this is maximized when $\boldsymbol{x}$ is along the vector $(p\boldsymbol{x}_0 - (1-p)\boldsymbol{x}_1)$ and also when the norm of $\boldsymbol{x}$ is maximised (i.e. $\|\boldsymbol{x}\| = 1$ as it must be in the Bloch sphere). So, M_0 is the projector onto quantum state with bloch vector $\boldsymbol{x} = \frac{p\boldsymbol{x}_0 - (1-p)\boldsymbol{x}_1}{\|p\boldsymbol{x}_0 - (1-p)\boldsymbol{x}_1\|}$ and M_1 is the quantum state with Bloch vector $-\boldsymbol{x}$. In this case, the optimal probability for a successful guess is

$$p_{guess} = \frac{1}{2}\left(1 + \|p\boldsymbol{x}_0 - (1-p)\boldsymbol{x}_1\|\right) \, .$$

2. Show that the Schatten p-norms are unitarily invariant. That is, $||A||_p = ||UAV||_p$ for any unitaries U and V.

   Answer 2: The Schatten p-norm of a matrix A can be written in terms of its singular values, $\sigma_i(A)$, as follows

$$||A||_p = \left(\sum_i \sigma_i^p(A)\right)^{1/p} \, . \quad (4)$$

   One properties of the singular values is that for any unitaries U and V, we have $\sigma_i(A) = \sigma_i(UAV)$. Indeed, assuming that the singular value decomposition of A is given by $A = W_1 \Sigma W_2^*$ where W_1 and W_2 are (left and right) unitaries and Σ is the rectangular diagonal matrix containing the singular values. Thus, we have that the singular value decomposition of UAV is given by $UAV = (UW_1)\Sigma(W_2^* V)$ where (UW_1) and $(V^* W_2)$ respectively the new left and right unitaries that perform the SVD, but the rectangular diagonal matrix Σ is unchanged (i.e. same singular values).

3. Write $\|Q\|_\infty$ in terms of the eigenvalues of Q (you may assume Q is square).
Answer 3: We have $\|Q\|_\infty = \max_i[\sigma_i(Q)]$ (i.e. the largest singular value of Q).

4. Use the unitary invariance of the Schatten p-norms and the triangle inequality to show that
$$||U^N - V^N||_p \leqslant N||U - V||_p\,.$$
Answer 4: We can use telescoping series to rewrite $U^N - V^N$ as follows
$$U^N - V^N = U\left(\sum_{k=0}^{N-1} U^{N-1-k}V^k\right) - \left(\sum_{k=0}^{N-1} U^{N-1-k}V^k\right)V \tag{5}$$
$$= \sum_{k=0}^{N-1}\left(U^{N-k}V^k - U^{N-1-k}V^{k+1}\right)\,. \tag{6}$$
So, now we can use triangle inequality and unitary invariance to write
$$||U^N - V^N||_p \leqslant \sum_{k=0}^{N-1} ||U^{N-k}V^k - U^{N-1-k}V^{k+1}||_p \tag{7}$$
$$= \sum_{k=0}^{N-1} ||U - V||_p \tag{8}$$
$$= N||U - V||_p\,, \tag{9}$$
where the first inequality is obtained from triangle inequality and the first equality is obtained from unitary invariance by multiplicating, for each k, the operator inside the norm by $(U^\dagger)^{N-1-k}$ on the left and by $(V^\dagger)^k$ on the right i.e. from unitary invariance we have, for each k, the equality
$$||U^{N-k}V^k - U^{N-1-k}V^{k+1}||_p = ||(U^\dagger)^{N-1-k}(U^{N-k}V^k - U^{N-1-k}V^{k+1})(V^\dagger)^k||_p = ||U - V||_p\,.$$

5. Show that $\|Q\|_1 \geqslant \|Q\|_2 \geqslant \|Q\|_\infty$.
Answer 5: From the definition of the schatten norm, Eq. (4), we have
$$||Q||_1^2 = \left(\sum_i \sigma_i(Q)\right)^2 \tag{10}$$
$$= \sum_i \sigma_i^2(Q) + \sum_{i\neq j}\sigma_i(Q)\sigma_j(Q) \tag{11}$$
$$\geqslant \sum_i \sigma_i^2(Q) \tag{12}$$
$$= ||Q||_2^2\,, \tag{13}$$
where in the second equality we simply separated the square terms and cross terms. In the inequality, we have used the positivity of the singular values. So, by positivity of the norm we have $||Q||_1 \geqslant ||Q||_2$. Now using the results from exercise 3 (and by positivity of the singular values) obviously we have
$$||Q||_2^2 = \sum_i \sigma_i^2(Q) \tag{14}$$
$$\geqslant \max_i[\sigma_i^2(Q)] \tag{15}$$
$$= \left(\max_i[\sigma_i(Q)]\right)^2 \tag{16}$$
$$= ||Q||_\infty^2\,, \tag{17}$$
which ends the proof.

6. Now for any $p, q \geqslant 1$, show that $\|Q\|_p \leqslant \|Q\|_q \iff p \geqslant q$. Notice that $p > q$ does not imply that $\|Q\|_p > \|Q\|_q$. (Hint: you can use the fact that for real positive x_i and $r \geqslant 0$, we have $\sum_i x_i^{1+r} \leqslant \left(\sum_i x_i\right)^{1+r}$.)

   Answer 6: For $p \geqslant q$, we can define $p/q = 1 + r$ where $r \geqslant 0$ and use the hint as follows

$$\|Q\|_q^p = \left(\sum_i \sigma_i^q(Q)\right)^{p/q} \tag{18}$$

$$\geqslant \sum_i \sigma_i^p(Q) \tag{19}$$

$$= \|Q\|_p^p\,, \tag{20}$$

   so as $p \geqslant 1$ we have $\|Q\|_q^p \geqslant \|Q\|_p^p \implies \|Q\|_q \geqslant \|Q\|_p$.

7. Write both python and mathematica code that can compute the Schatten 1, 2, and infinity norms of a (Haar) random unitary matrix. What is the average distance (for each norm) between two randomly chosen unitaries. **First answer (analytical part):** For any unitary, the singular values are 1, so you should obtain $\|U\|_p = d^{1/p}$. Thus respectively for Schatten 1, 2, and infinity norms we have d, $\sqrt{d}$ and 1.