Solutions Problem Sheet 10: Entropy

Text Book

Many solutions in this problem sheet can be found in the standard text books. We will refer to some parts of "Quantum Computation and Quantum Information" 10th edition by Michael A. Nielsen and Isaac L. Chuang, which can be found online for free here, or by googling "Quantum Computation and Quantum Information".

Class problems

1. Compute the entropy of the full state and reduced states of:

   a) $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$

   b) $\rho = 3/7|00\rangle\langle 00| + 4/7|++\rangle\langle ++|$

   c) $\rho = 3/7|00+\rangle\langle 00+| + 4/7|++-\rangle\langle ++-|$

   Answers 1: For a), the full density matrix is already expressed in a diagonal form in the Bell basis with eigenvalues $\lambda_1 = 3/4$, $\lambda_2 = 1/4$ and zero for the rest. Thus, the entropy of the full state is

$$S(\rho) = -\sum_i \lambda_i \log(\lambda_i) = -3/4\log(3/4) - 1/4\log(1/4) \approx 0.811 \ . \tag{1}$$

   The reduced state to any qubit is a maximally mixed state of one qubit i.e., $\rho_1 = \rho_2 = \mathbb{1}/2$, leading to

$$S(\rho_1) = S(\rho_2) = -1/2\log(1/2) - 1/2\log(1/2) = 1 \ . \tag{2}$$

   For b), the density matrix is not diagonalized yet and we have to do that (as $|00\rangle$ and $|++\rangle$ are not orthogonal the eigenvalues are not $\{3/7, 4/7\}$ as in c)). The density matrix can be written out explicitly as

$$\rho = \begin{pmatrix} 4/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 \end{pmatrix} \tag{3}$$

   . Diagonalizing this ρ by hands is really painful, so please use Mathematica to do it. By doing so, the eigenvalues are found to be $\{a = (7+\sqrt{13})/14, 1-a = (7-\sqrt{13})/14, 0, 0\}$, leading to the entropy of $S(\rho) \approx 0.799$. The reduced state to any of the qubits is $\rho_1 = \rho_2 = 3/7|0\rangle\langle 0| + 4/7|+\rangle\langle +|$, which has eignevalues of $\{6/7, 1/7\}$ and corresponding entropy of around 0.592.

   For c), As $\langle 00+|++-\rangle = 0$, the density matrix is *already* in the diagonal form with non-zero eigenvalues 3/7 and 4/7, leading to the entropy of $S(\rho) = -(3/7)\log(3/7) - (4/7)\log(4/7) \approx 0.985$. Now, we want to look at the subsystems. By tracing out the last qubit, we come back to the question b). For other subsystems contains two qubits, the entropy is the same as the full state – can you see why ?

2. Prove that the relative entropy $S(\rho||\sigma) \geqslant 0$. (Hint - you can use that for any doubly stochastic matrix P, concave function f and probability distribution $\boldsymbol{q}$ we have that $\sum_j P_{ij} f(q_j) \leqslant f(p_i)$, where $p_i = \sum_j P_{ij} q_j$.)

   Answers 2: See the proof in p. 511 and 513 in the Nielsen and Chuang's book. Notice that the inequality given here is simply Jensen's inequality which is also used to show the positivity of the KL in an easy way (you can also show that the KL is minimized iif $p(x) = q(x)$ and in this case the KL is 0).

$$
\begin{aligned}
-\sum_x p(x)\log(q(x)/p(x)) &= \mathbb{E}_{x\sim p}[-\log(q(x)/p(x)] && (4)\\
&\geqslant -\log(\mathbb{E}_{x\sim p}[q(x)/p(x)]) && (5)\\
&= 0\ . && (6)
\end{aligned}
$$

Notice that to show that the minimum is obtained iif $q(x) = p(x)$, you can use Lagrange multipliers method (i.e. minimize the KL w.r.t $\{q(x)\}$ with condition $\sum_x q(x) = 1$).

3. Show that a pure state $|\psi\rangle_{AB}$ is entangled between subsystems A and B if and only if the conditional entropy is less than zero i.e., $S(A|B) < 0$.

Answers 3: Consider the conditional entropy $S(A|B) = S(A,B) - S(B)$, where $S(A,B) = \mathrm{Tr}[\rho_{AB}\log(\rho_{AB})]$ and $S(B) = \mathrm{Tr}[\rho_B \log(\rho_B)]$ with $\rho_B = \mathrm{Tr}_A[\rho_{AB}]$ is a reduced state on a subsystem B. When ρ_{AB} is pure, then $S(\rho_{AB}) = 0$ and the expression is simplified to

$$S(A|B) = -S(B)\ , \tag{7}$$

If ρ_{AB} is entangled, we have ρ_B to be mixed and $S(B) > 0$, leading to $S(A|B) < 0$. This proves one direction of a 'if and only if' statement. To prove the other direction, if $S(A|B) < 0$, we have that $S(B) > 0$, implying that ρ_B is mixed. This happens only when ρ_{AB} is entangled, which completes the proof.

4. For a composite system AB in a pure state $|\psi\rangle_{AB}$, show that $S(\rho_A) = S(\rho_B)$ (Hint - think about the Schmidt decomposition.)

Answers 4: When the composite system is in a pure state, by the Schmidt decomposition both subsystems A and B share the same eigenvalues, which implies that they have the same entropy.

5. Prove the triangle inequality: $|S(\rho_A) - S(\rho_B)| \leqslant S(\rho_{AB}) \leqslant S(\rho_A) + S(\rho_B)$ (Hint - for the first inequality, consider purification and, for the second inequality, consider the result of Question 2).

Answers 5: See the proof in p. 515-516 in the Nielsen and Chuang's book. (We first prove the second inequality which is obtained using $H(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B)$ and Question 2 (which states that $S(\rho_{AB}\|\rho_A\otimes\rho_B) \geqslant 0$). The first inequality is obtained by considering the purification of ρ_{AB}, the results of Question 4 and the second inequality.)

6. Verify that $H(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B)$, and show that discarding a quantum system can never increase mutual information i.e., $H(AB:C) \geqslant H(B:C)$. (Hint - consider the data-processing inequality)

Answers 6: First let us prove that $\log(\rho_A \otimes \rho_B) = \log(\rho_A) + \log(\rho_B)$. Consider the eigendecomposition of ρ_A and ρ_B as $\rho_A = \sum_i \alpha_i |\alpha_i\rangle\langle\alpha_i|$ and $\rho_B = \sum_i \beta_i |\beta_i\rangle\langle\beta_i|$. Note that $\rho_A \otimes \rho_B = \sum_{i,j} \alpha_i\beta_j |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_j\rangle\langle\beta_j|$ is already in the diagonal form. So, we have

$$
\begin{aligned}
\log(\rho_A\otimes\rho_B) &= \log\left(\sum_{i,j}\alpha_i\beta_j|\alpha_i\rangle\langle\alpha_i|\otimes|\beta_j\rangle\langle\beta_j|\right) && (8)\\
&= \sum_{i,j}\log(\alpha_i\beta_j)|\alpha_i\rangle\langle\alpha_i|\otimes|\beta_j\rangle\langle\beta_j| && (9)\\
&= \sum_{i,j}\left(\log(\alpha_i)+\log(\beta_j)\right)|\alpha_i\rangle\langle\alpha_i|\otimes|\beta_j\rangle\langle\beta_j| && (10)\\
&= \sum_i \log(\alpha_i)|\alpha_i\rangle\langle\alpha_i| + \sum_j \log(\beta_j)|\beta_j\rangle\langle\beta_j| && (11)\\
&= \log(\rho_A)\otimes\mathbb{1}_B + \mathbb{1}_A\otimes\log(\rho_B)\ . && (12)
\end{aligned}
$$

We are ready to verify the expression.

$$\begin{aligned} S(\rho_{AB}\|\rho_A\otimes\rho_B) =& \mathrm{Tr}[\rho_{AB}\log(\rho_{AB})] - \mathrm{Tr}[\rho_{AB}\log(\rho_A\otimes\rho_B)] && (13)\\ =& - S(\rho_{AB}) - \mathrm{Tr}\left[\rho_{AB}\left(\log(\rho_A)\otimes \mathbb{1}_B + \mathbb{1}_A\otimes\log(\rho_B)\right)\right] && (14)\\ =& - S(\rho_{AB}) - \mathrm{Tr}_A\left[\mathrm{Tr}_B[\rho_{AB}]\log(\rho_A)\right] - \mathrm{Tr}_B\left[\mathrm{Tr}_A[\rho_{AB}]\log(\rho_B)\right] && (15)\\ =& - S(\rho_{AB}) + S(\rho_A) + S(\rho_B) && (16)\\ =& H(A:B)\ , && (17) \end{aligned}$$

where in the third equality we used $\mathrm{Tr}_A[Q_{AB}(\mathbb{1}_A\otimes O_B)] = \mathrm{Tr}_A[Q_{AB}]O_B$ (you can easily check it). Now, we can write $H(AB:C) = S(\rho_{ABC}\|\rho_{AB}\otimes\rho_C)$ and $H(B:C) = S(\rho_{BC}\|\rho_B\otimes\rho_C)$. Recall the entropy version of data-processing inequality. For quantum states ρ and σ as well as a channel $\mathcal{E}(\cdot)$, we have

$$S(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leqslant S(\rho\|\sigma)\ . \tag{18}$$

In this case, we have the channel to be a partial trace over the system A i.e., $\mathcal{E}(\cdot) = \mathrm{Tr}_A[\cdot]$ and $\rho = \rho_{ABC}$ and $\sigma = \rho_{AB}\otimes\rho_C$. Plugging these into the data-processing inequality leads to the desired result.