Problem Sheet 12: Pure state bipartite entanglement

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

Resource Theories.

1. a) Argue that any resource theory defines a partial order $\rho \preceq \sigma$ on the set of all quantum states $\mathcal{D}$ based on whether or not we can send $\rho \to \sigma$ under some free operation $\mathcal{E} \in F$.

   b) How can the notion of a resource measure M be defined using this partial ordering?

   c) Show that any two free states σ_1 and σ_2 are equal under the partial ordering.

   d) Show that any resource measure M must ascribe the same value to any two states that are equal under this ordering. (Does the converse hold?)

2. Make up your own resource theory! (Optional)

3. Prove that it is possible for Bob to perform a local operation conditional on an outcome of Alice's measurement via LOCC. That is, show it is possible to perform $\rho_{AB} \to \sum_i (M_i \otimes U_i)\rho_{AB}(M_i^\dagger \otimes U_i^\dagger)$ via LOCC.

4. Prove that product states are the states that can be prepared freely via LOCC.

Majorization.

1. Draw the partial order defined by majorization on the following vectors $\boldsymbol{v}_1 = (0, 1/3, 2/3)$, $\boldsymbol{v}_2 = (1/3, 1/3, 1/3)$, $\boldsymbol{v}_3 = (1/4, 1/5, 1/3)$, $\boldsymbol{v}_4 = (1/5, 3/5, 1/5)$, $\boldsymbol{v}_5 = (2/5, 1/2, 1/10)$, and $\boldsymbol{v}_6 = (0, 1, 0)$.

2. Show that $(1/N, 1/N, ..., 1/N) \preceq \boldsymbol{p} \preceq (1, 0, ..., 0)$ for any probability distribution $\boldsymbol{p}$.

3. A useful equivalent definition of majorization is that $\boldsymbol{x} \preceq \boldsymbol{y}$ iff $\boldsymbol{x}$ is a convex combination of vectors obtained by permuting coordinates of $\boldsymbol{y}$.

   Use this fact to show that the diagonal elements of a density operator are majorized by its eigenvalues. (Super useful property!)

   (Hint use the fact that any doubly stochastic matrix can be written as a convex combination of permutation matrices).

Bipartite Entanglement.

1. Argue that $|\psi_-\rangle_{AB}$ can be transformed into any state $|\phi\rangle_{AB}$ via LOCC using Nielson's Majorization Theorem. Describe a protocol to do this in practise.

2. Show that transforming between the states $|\phi\rangle = \sqrt{\frac{15}{100}}|00\rangle + \sqrt{\frac{3}{10}}|11\rangle + \sqrt{\frac{4}{10}}|22\rangle + \sqrt{\frac{15}{100}}|33\rangle$ and $|\psi\rangle = \sqrt{\frac{3}{10}}|00\rangle + \sqrt{\frac{3}{10}}|11\rangle + \sqrt{\frac{3}{10}}|22\rangle + \sqrt{\frac{1}{10}}|33\rangle$ is not possible deterministically via LOCC.

Assessed Problem

1. *Majorization being useful.*

   The aim of this question is to prove that $\arg\min_U ||\rho - U\sigma U^\dagger||_2^2$ is the unitary that rotates σ to the eigenbasis of ρ.

   a) Show that minimizing $||\rho - U\sigma U^\dagger||_2^2 = \mathrm{Tr}[(\rho - U\sigma U^\dagger)^2]$ is equivalent to maximising $\mathrm{Tr}[\rho\sigma_U]$ where $\sigma_U = U\sigma U^\dagger$.

   b) Show that $\mathrm{Tr}[\rho\sigma] = \boldsymbol{\lambda}.\boldsymbol{s}$ where $\boldsymbol{\lambda}$ are the eigenvalues of ρ (listed in decreasing order) and $\boldsymbol{s}$ are the diagonal elements of σ in the ordered eigenbasis of ρ.

   For any convex function f we have that $f(\boldsymbol{x}) \leqslant f(\boldsymbol{y})$ iff $\boldsymbol{x} \prec \boldsymbol{y}$ (super useful property!).

   c) Use this, and the fact that the dot product is a convex function, to show that the unitary that minimizes $||\rho - U\sigma U^\dagger||_2^2$ is the unitary that rotates σ to the eigenbasis of ρ.

2. *Interconversion via LOCC.*

   a) State the necessary and sufficient condition for the deterministic transformation of $|\phi\rangle_{AB}$ to $|\psi\rangle_{AB}$ via LOCC.

   b) What deterministic LOCC transformations (one way, two way or no-way) are possible between the following states where both A and B are $d = 6$ dimensional Hilbert space:

   i) $|\phi\rangle = \sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{6}}(|11\rangle + |55\rangle)$ and $|\psi\rangle = \sqrt{\frac{1}{2}}(|01\rangle + |10\rangle)$

   ii) $|\phi\rangle = \sqrt{\frac{1}{2}}|00\rangle + \sqrt{\frac{2}{5}}|12\rangle - e^{i\pi/3}\sqrt{\frac{1}{10}}|55\rangle$ and $|\psi\rangle = \sqrt{\frac{3}{5}}|00\rangle - \sqrt{\frac{1}{5}}|12\rangle + \sqrt{\frac{1}{5}}|22\rangle$

   iii) $|\phi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$ and any state $|\psi\rangle$. State at least a specific condition on $|\psi\rangle$ such that $|\phi\rangle \underset{LOCC}{\rightarrow} |\psi\rangle$ (is it true for any $|\psi\rangle$?).

   c) Construct an explicit LOCC protocol (measurements, unitaries, communications) for one of the previous examples.

   d) Briefly argue (1-2 points) that classical physics corresponds to a proper subset of LOCC.