Problem Sheet 11: Fidelity

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. Compute the fidelity between the following pairs of states:

   a) $\rho = |\Phi_+\rangle$ and $\sigma = |00\rangle$

   b) $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$ and $\sigma = |00\rangle$.

   c) $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$ and $\sigma = 3/7|00\rangle\langle 00| + 4/7|++\rangle\langle ++|$

2. We will keep using this polar decomposition throughout this class problem. So, here is some kick-start for you to prove it.

   a) Show that any complex square matrix A can be expressed in the polar decomposition $A = |A|U$ where $|A| = \sqrt{AA^\dagger}$ and U is some unitary (Hint - singular value decomposition).

   b) Show that $\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \sqrt{\rho}\sqrt{\sigma}\tilde{V}$ for some unitary $\tilde{V}$.

3. This question works you through the proof of Uhlmann's theorem.

   a) Use Cauchy–Schwarz inequality to show that $\mathrm{Tr}[AU] \leqslant \mathrm{Tr}|A|$ where we write A in terms of its polar decomposition as $A = |A|V$. In addition, verify that equality is attained with $U = V^\dagger$.

   b) Why can the purification of a state ρ be written as $|\psi\rangle = U_R \otimes \sqrt{\rho}U_S|\mathrm{Vec}(\mathbb{I})\rangle$? Where U_S and U_R are unitaries. (Hint: Schmidt decomposition).

   c) Show that $\langle\mathrm{Vec}(\mathbb{I})|A^* \otimes B|\mathrm{Vec}(\mathbb{I})\rangle = \mathrm{Tr}[A^\dagger B]$

   d) Hence show that $\max_{|\psi\rangle,|\phi\rangle} |\langle\psi|\phi\rangle| = \mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}] = \mathrm{F}(\rho,\sigma)$ where the maximisation is taken over all purifications $|\psi\rangle$ and $|\phi\rangle$ of ρ and σ.

4. (optional) Use Uhlmann's theorem to show that the data processing inequality holds for quantum fidelity, i.e. $\mathrm{F}(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \geqslant \mathrm{F}(\rho,\sigma)$ for any trace preserving quantum operation $\mathcal{E}$.

5. This question works you through the derivation of the following operational expression for the fidelity

$$F(\rho,\sigma) = \mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}] = \min_{\{M_i\}} \sum_i \sqrt{\mathrm{Tr}[\rho M_i]\mathrm{Tr}[\sigma M_i]}\,. \tag{1}$$

   a) Show that $\mathrm{Tr}[\sqrt{\rho^{1/2}\sigma\rho^{1/2}}] = \sum_i \mathrm{Tr}[\sqrt{\rho}\sqrt{M_i}\sqrt{M_i}\sqrt{\sigma}V]$ for a set of POVMs $\{M_i\}$ where V is some unitary (Hint - polar decomposition).

   b) Hence use Cauchy-Schwarz to show Eq. (1).