Problem Sheet 4: Quantum Channels (Part 1) and Naimark's Theorem

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. Consider the matrices:

$$A_0 \propto \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A_1 \propto \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{1}$$

a) Show that A_0 and A_1 are Kraus operators of a quantum operation given the right normalization factor.

b) What are the dimensions of the input quantum system and the output quantum system?

2. Suppose we have a qubit system A interacting with a qubit environment through the unitary

$$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes X_B \tag{2}$$

a) Write down the channel induced on the system assuming that the environment qubit starts in the state $|0\rangle$.

Suppose now

$$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes e^{-i\lambda t X_B} \ ? \tag{3}$$

b) Write down the channel induced on the system in this case.

c) Comment on what this tells you about the link between system and environment interactions and decoherence.

3. Suppose we have a qubit system A interacting with a qubit environment through the unitary

$$U = \frac{1}{\sqrt{2}} \left(X_A \otimes \mathbb{I}_B + Y_A \otimes X_B \right) . \tag{4}$$

a) What is the channel induced on the system in this case assuming that the environment starts in the state $|0\rangle$.

b) By looking at the action of the channel on a Bloch vector describe the action of this channel.

4. *Question 4d) from the measurement problem sheet.* Suppose Bob hands you a quantum state and promises that it is either $|\phi_1\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ or $\cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$.

A measurement that can distinguish these states perfectly some of the time (and is inconclusive at other times) is given by $\mathcal{M} = \{M_0, M_1\}$ with

$$\begin{aligned} M_0 &= \frac{|\cos(\theta)| + \cos(\theta)}{1 + |\cos(\theta)|}|0\rangle\langle 0| + \frac{|\cos(\theta)| - \cos(\theta)}{1 + |\cos(\theta)|}|1\rangle\langle 1| \\ M_1 &= \frac{1}{1 + |\cos(\theta)|}|\phi_1^{\perp}\rangle\langle\phi_1^{\perp}|, M_2 = \frac{1}{1 + |\cos(\theta)|}|\phi_2^{\perp}\rangle\langle\phi_2^{\perp}| \end{aligned} \tag{5}$$

d) Write down a projective measurement on a larger system that can be used to realise this POVM.

5. Optional: Take a look at *Ancilla-free implementation of generalized measurements for qubits embedded in a qudit space* (Phys. Rev. Research 4, 033027 (2022)).

   a) Why and how does this paper use ‘informationally complete’ POVMs?

   b) How is Naimark’s theorem used to realize these measurements?

   c) Discuss this paper (no need to write anything down). Do you think it’s any good? Any ideas about how you might like to extend it?

   d) Got time to kill? Have a go at implementing their proposal!

Assessed Problem

(Short answer question from Exam 2024)

Consider a measure-and-update channel $\mathcal{E}(\rho) = \sum_k |k\rangle\langle k| \otimes B_k \rho B_k^\dagger$ where ρ is the initial state to be measured, $\{|k\rangle\}$ is an orthonormal basis and $\{B_k\}$ are a set of Kraus operators.

1. Write down the Kraus operators for the channel $\mathcal{E}$. Explain why this is called the 'Measure-and-update' channel.

Now consider the following choice in operators $\{B_k\}$:

$$B_1 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

where λ is real and $0 \leqslant \lambda \leqslant 1$.

Further suppose the initial state is the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

2. What is the probability of measuring outcomes 1 and 2? What are the corresponding output states of the system that is measured? (5 marks)

Suppose you now instead have the operators $\{B_k'\}$:

$$B_1' = \begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}, \quad B_2' = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

where λ is defined above.

3. What are the probabilities and output states in this case?
4. What do you conclude from comparing your answers in (b) and (c)?