Problem Sheet 9: Distinguishing quantum states and Matrix norms

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. Suppose you are given a state ρ_0 with probability p and a state ρ_1 with probability $1-p$ (single qubit states).

   a) What is the optimum binary POVM to answer the question 'Is the system in the state ρ_0 or ρ_1'?

   b) What is the probability of successfully distinguishing ρ_0 and ρ_1?

2. Show that the Schatten p-norms are unitarily invariant. That is, $||A||_p = ||UAV||_p$ for any unitaries U and V.

3. Write $\|Q\|_\infty$ in terms of the eigenvalues of Q (you may assume Q is square).

4. Use the unitary invariance of the Schatten p-norms and the triangle inequality to show that

$$||U^N - V^N||_p \leqslant N||U - V||_p \,.$$

5. Show that $\|Q\|_1 \geqslant \|Q\|_2 \geqslant \|Q\|_\infty$.

6. Now for any $p, q \geqslant 1$, show that $\|Q\|_p \leqslant \|Q\|_q \iff p \geqslant q$. Notice that $p > q$ does not imply that $\|Q\|_p > \|Q\|_q$. (Hint: you can use the fact that for real positive x_i and $r \geqslant 0$, we have $\sum_i x_i^{1+r} \leqslant \left(\sum_i x_i\right)^{1+r}$. If you use it, try at least to show it.).

7. Write (/find in built functions) in Python and Mathematica code that can compute the Schatten 1, 2, and infinity norms of a (Haar) random unitary matrix. What is the average distance (for each norm) between two randomly chosen unitaries.

Assessed Problem

Say you have some arbitrary channel $\mathcal{E}$ and you want to bound $||\mathcal{E}(O)||_2$ where O is an arbitrary observable. You can assume that $\mathcal{E}$ maps a d_S dimensional system to a d_S dimensional system.

We will assume that this channel can be written in Kraus form as $\mathcal{E}(...) = \sum_{k=1}^{M} B_k ... B_k^\dagger$.

You can also assume that the channel has the Stinespring dilation using a d_E dimensional ancilla system.

1. Derive the tightest upper bound you can for $||\mathcal{E}(O)||_2$ in terms M, d_S, and the norm of O (you can pick the norm).

   Hint- you may want to derive multiple bounds depending on whether O is a projector or a (sum of) Pauli operators.

2. Derive the tightest possible upper bound you can for $||\mathcal{E}(O)||_2$ which is independent of M. It may, or may not, also depend on d_E and/or d_S. It will depend on the norm of O (you can pick the norm).

   Hint- you may want to derive multiple bounds depending on whether O is a projector or a (sum of) Pauli operators.

3. Discuss which of these bounds is tighter?

4. What is the tightest bound you can obtain on $||\mathcal{E}^*(P)||_2$ where $\mathcal{E}^*$ is the adjoint of a channel and P is a Pauli? (Note: the adjoint of a channel is not necessarily a channel.)