Problem Sheet 3

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

Purifications

1. Compute purifications for the following states:

   a) $\rho_1 = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$
   b) $\rho_2 = 1/2(|0\rangle\langle 0| + |+\rangle\langle +|)$
   c) $\rho_3 = 1/2(|\psi_+\rangle\langle\psi_+| + |\phi_-\rangle\langle\phi_-|)$

2. Consider the single qubit state $\rho = \frac{1}{2}(\mathbb{I} + 0.1X + 0.1Y + 0.2Z)$

   a) Write ρ as a matrix in the computational basis.

   b) Compute the eigen-decomposition of ρ.

   c) Is ρ mixed or pure? How do you know?

   d) Compute a pure state decomposition of ρ involving three states.

   e) Hence state i. a purification of ρ using a single qubit environment and ii. a purification using a qutrit environment.

Measurements

3. Explain what is the difference between POVM measurements, projective measurements and a measurement of an observable.

4. a) Write down a POVM measurement $\mathcal{M}$ that asks the question: "Is the system in the $|\Phi_+\rangle$ Bell state?"

   b) Consider the 2-qubit separable state $|\psi\rangle \otimes |\psi\rangle$. What is the probability to find the system in $|\Phi_+\rangle$?

5. Consider a d dimensional system S and $\mathcal{M} = \{\alpha_0\mathbb{I}, \alpha_1\mathbb{I}, \alpha_2\mathbb{I}, \alpha_3\mathbb{I}\}$. Is this a valid measurement? If so, what does the measurement do?

6. Suppose Bob hands you a quantum state and promises that it is either $|\phi_1\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ or $\cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$.

   a) Sketch these states on the Bloch sphere.

   b) Design a measurement that perfectly distinguishes the states *some* of the time, is inconclusive at others, but never makes a mistake.

   c) What is the probability in a *single* run of the experiment that you guess correctly?

   d) Write down a projective measurement on a larger system that can be used to realise this POVM.

7. Propose an 'informationally complete' measurement for a single qubit state. That is, a POVM measurement $\mathcal{M}$ that allows you to perfectly reconstruct a single qubit quantum state. What about a 2-qubit state?

8. Suppose $\mathcal{M} = \{M_i\}_{i=1}^m$ and $\mathcal{N} = \{N_i\}_{i=1}^n$ are two different POVM measurements.

   a) We can define an $m + n$ outcome POVM from $\mathcal{M}$ and $\mathcal{N}$ by flipping a biased coin and with probability p doing $\mathcal{M}$ and probability $(1 - p)$ doing $\mathcal{N}$. Write down the measurement operators for this measurement.

   b) Suppose now $m = n$. We can alternatively define a m measurement composed of the operators $\{pM_i + (1 - p)N_i\}_{i=1}^m$. How does this measurement differ from the one in part (a)? How could you realise it?

Assessed Problems

1. (Canonical purification) Let ρ_A be a density operator and let $\sqrt{\rho_A}$ be its unique positive semi-definite square root (i.e., $\rho_A = \sqrt{\rho_A}\sqrt{\rho_A}$.) We define the canonical purification of ρ_A as follows:

$$|\psi^\rho\rangle_{RA} := (I_R \otimes \sqrt{\rho_A})|\Omega\rangle_{RA}\,,$$

where $\dim(R) = \dim(A)$ and $|\Omega\rangle_{RA} = \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_A$ is the unnormalized maximally entangled vector. Show that $|\psi^\rho\rangle_{RA}$ is a purification of ρ_A.

2. In this problem we will learn the concept of convexity and extreme points of operators. We will show that pure states are extreme points of the convex set of states and orthogonal measurements are extreme points of the convex set of 2-outcome POVMs. We start with the definition of extreme points.

   Consider the space of bounded operators $\mathcal{B}$, for any operators A, O_1, O_2, we say that A ***lies between*** O_1 and O_2 if $O_1 \neq O_2$ and there exists a $0 < p < 1$ such that $A = pO_1 + (1-p)O2$. If $\mathcal{H}$ is subspace of $\mathcal{B}$ and $A \in \mathcal{H}$, we call A is ***an extreme point*** of $\mathcal{H}$ if it does not lie between any two distinct points of $\mathcal{H}$. That is $A = pO_1 + (1-p)O2$ if and only if $A = O_1$ $(p = 1)$ or $A = O_2$ $(p = 0)$.

   a) Show that extreme points of the set of quantum states are pure states, and pures states are extreme points.

   Let $F = \{F_1, F_2\}$ and $G = \{G_1, G_2\}$ be two POVMs. We define an element-wise convex combination of F and G as $\alpha F + (1-\alpha)G := \{\alpha F_1 + (1-\alpha)G_1, \alpha F_2 + (1-\alpha)G_2\}$, with $0 \leqslant \alpha \leqslant 1$.

   b) Consider a POVM with two outcomes and respective measurement operators E and $\mathbb{1} - E$. Suppose that E has an eigenvalue λ such that $0 < \lambda < 1$. Show that the POVM is not extremal by expressing it as a nontrivial convex combination of two POVMs.
   Hint: Consider the spectral decomposition of E and rewrite it as a convex combination of two POVM elements.

   c) Suppose that E is an orthogonal projector. Show that the POVM cannot be expressed as a nontrivial convex combination of POVMs.

   d) What is the operational interpretation of an element-wise convex combination of POVMs?