Problem Sheet 2: Quantum Basics

Don't initially use mathematica for this! (But maybe use it to double check your answers).

Quantum Theory

1. The state space of a single qubit can be represented by a Bloch vector $\boldsymbol{r}$ with norm less than 1, i.e. $|\boldsymbol{r}| \leqslant 1$. A pure state is a state that cannot be written in the form $\rho = p\sigma_0 + (1-p)\sigma_1$ for any two states σ_0, σ_1 and any $0 < p < 1$.

   a) Argue (geometrically!) that any pure state has a Bloch vector of norm 1 and hence can be written as $|\psi\rangle\langle\psi|$.

   b) Sketch on the Bloch sphere the states: $|1\rangle$, $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|+y\rangle := \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, and $\frac{1}{\sqrt{5}}(|0\rangle + i\sqrt{4}|1\rangle)$.

   c) $\mathrm{Tr}[\rho^2]$ is known as the purity of a state. Argue why this is an appropriate name.

2. Are the following states pure or mixed?
   a) $\rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$
   b) $\rho = \frac{1}{6}(3|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + 3|1\rangle\langle 1|)$

3. a) Sketch the points $\frac{1}{5}|0\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|$, $\frac{1}{3}|-y\rangle\langle -y| + \frac{2}{3}|0\rangle\langle 0|$ on the Bloch sphere.
   (A geometric answer will suffice I don't need the exact messy algebra).

   b) Give a geometric argument to show that $\frac{1}{2}\mathbb{I}/2$ can be written as a convex combination of an infinite number of pairs of states.

   c) Write $\frac{1}{2}\mathbb{I}$ as a convex combination of 4 states (easy). How about as a convex combination of three states?

4. Given any quantum state ρ of a d-dimensional quantum system S, explain why we can write $\sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ where λ_k are real and positive and $\langle\lambda_k|\lambda_j\rangle = \delta_{jk}$. What does this tell us about how we can interpret ρ?

Partial trace and unitaries

1. Compute the partial trace ρ_A of the following states:

   a) $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
   b) $|\psi\rangle_{AB} \propto (\alpha|0+\rangle + \beta|11\rangle)$
   c) $|\psi\rangle_{AB} \propto (|0+\rangle + 3|\psi 1\rangle + 5i|2-\rangle))$, where $|\psi\rangle$ is a state on A.

2. Optional! If you've done this before and are happy with it feel free to skip.

   a) Use a series expansion to show that $e^{-i\theta\sigma_i/2} = \cos(\theta/2)\mathbb{1} - i\sin(\theta/2)\sigma_i$.

b) What is the effect of evolving the state $|+\rangle$ under $e^{-i\theta\sigma_z/2}$ for $\theta = \pi/2$? State the final state and sketch this evolution on the Bloch sphere.

c) What about evolving $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$ under $e^{-i\theta\sigma_x/2}$ for $\theta = -3\pi/4$?