# Problem Sheet 10: Entropy

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

You can find all these proofs online but you'll learn more if you have a go at doing them yourself first.

**Class problems**

1. Compute the entropy of the full state and reduced states of:

   a) $\rho = 3/4|\Phi_+\rangle\langle\Phi_+| + 1/4|\Psi_-\rangle\langle\Psi_-|$

   b) $\rho = 3/7|00\rangle\langle 00| + 4/7|++\rangle\langle ++|$

   c) $\rho = 3/7|00+\rangle\langle 00+| + 4/7|++-\rangle\langle ++-|$

2. Prove that the relative entropy $S(\rho||\sigma) \geqslant 0$. (Hint - you can use that for any doubly stochastic matrix $P$, concave function $f$ and probability distribution $\boldsymbol{q}$ we have that $\sum_j P_{ij} f(q_j) \leqslant f(p_i)$, where $p_i = \sum_j P_{ij}\, q_j$.)

3. Show that a pure state $|\psi\rangle_{AB}$ is entangled between subsystems $A$ and $B$ if and only if the conditional entropy is less than zero i.e., $S(A|B) < 0$.

4. For a composite system $AB$ in a pure state $|\psi\rangle_{AB}$, show that $S(\rho_A) = S(\rho_B)$ (Hint - think about the Schmidt decomposition.)

5. Prove the triangle inequality: $|S(\rho_A) - S(\rho_B)| \leqslant S(\rho_{AB}) \leqslant S(\rho_A) + S(\rho_B)$ (Hint - for the first inequality, consider purification and, for the second inequality, consider the result of Question 2).

6. Verify that $H(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B)$, and show that discarding a quantum system can never increase mutual information i.e., $H(AB:C) \geqslant H(B:C)$. (Hint - consider the data-processing inequality)

**Assessed Problem**

1. *No assessed problem this week but heads up that proofs of this sort are an easy thing to put in an exam!*