Problem Sheet 6: Quantum Channels (Part 3)

Class problems

1. Consider the other quantum channels from last week's problem sheet. These were

$$\mathcal{E}(\rho) = p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z \tag{1}$$

the channel associated with the non-normalization Kraus operators

$$A_0 \propto \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A_1 \propto \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{2}$$

and the channel induced on the system quantum A by the unitary

$$U = \frac{1}{\sqrt{2}}\left(X_A \otimes \mathbb{I}_B + Y_A \otimes X_B\right) . \tag{3}$$

assuming the environment qubit starts in the state $|0\rangle$.

a) Write out the Choi matrices for these three channels.

b) Hence (or otherwise) find a (different) set of Kraus operators to represent the same channels.

c) Consider the operation

$$\mathcal{E}(\rho) = 1/3(\alpha\,\mathrm{Tr}[\rho]\mathbb{I} + \beta\,\rho^T)\,.$$

For what values of α and β does this operation preserve the trace (i.e. $\mathrm{Tr}[\rho] = \mathrm{Tr}[\mathcal{E}(\rho)]$)? For what values is it completely positive?

d) For the case where $\mathcal{E}$ represents a genuine quantum channel state a minimal Kraus representation for the channel.

e) Hence state a more general expression for any set of Kraus operators that can represent

2. a) Prove that a linear map $\mathcal{E}$ is completely positive iff $J(\mathcal{E}) = \mathcal{E} \otimes \mathbb{1}(|vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|)$ is positive.

b) Hence show that i. the dephasing channel is completely positive but ii. the transpose operation is not.

Assessed Problem

Normal form of single qubit channel. In general, any single-qubit channel $\mathcal{N}$ can be written in the following *normal form*

$$\mathcal{N}(\cdot) = U\mathcal{N}'(V(\cdot)V^\dagger)U^\dagger, \tag{4}$$

where U, V are arbitrary single-qubit unitaries. The action of $\mathcal{N}'$ on the single-qubit Pauli matrices is given by

$$\mathcal{N}'(I) = I + t_X X + t_Y Y + t_Z Z, \tag{5}$$
$$\mathcal{N}'(X) = D_X X, \tag{6}$$
$$\mathcal{N}'(Y) = D_Y Y, \tag{7}$$
$$\mathcal{N}'(Z) = D_Z Z, \tag{8}$$

where $\mathbf{D} = (D_X, D_Y, D_Z) \in [-1,1]^3$ and $\mathbf{t} = (t_X, t_Y, t_Z) \in [-1,1]^3$ are two vectors, which we refer as *normal form parameters*. We say that $\mathcal{N}$ is *unital* if $\mathcal{N}(I) = I$ and *non-unital* otherwise.

For simplicity here we assume $U = I = V$, and hence $\mathcal{N}(\cdot) = \mathcal{N}'(\cdot)$.

(a) Write down the general expression of the Choi state for the channel $\mathcal{N}$.

(b) If $\mathcal{N}$ is a depolarizing channel, then it can be written as $\mathcal{N}(\rho) = p\mathrm{Tr}[\rho]\frac{I}{2} + (1-p)\rho$ with $0 \leqslant p \leqslant 1$. Is this channel unital or non-unital?

(c) Decompose ρ using Bloch vector representation and explain the effect of the depolarizing channel in the Bloch sphere.

(d) Find the conditions on $\mathbf{D}$ and $\boldsymbol{t}$, such that $\mathcal{N}$ is a depolarizing channel.

(e) Find the Kraus operators for this channel? *Hint: Of course you can use Choi state to do this. But maybe it's easier to use the equality $\mathcal{I} = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z)$ you have proved in problem sheet 2.*

(f) Consider a channel such that $\mathbf{D} = (\sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma)$ and $\boldsymbol{t} = (0,0,\gamma)$ where $0 < \gamma \leqslant 1$. Is this channel unital or non-unital? What is the effect of this channel on a generic single qubit state? What happens if this channel is applied $N \to \infty$ times in succession?

(g) Find the Kraus operators corresponding to the channel with $\mathbf{D} = (\sqrt{1-\gamma}, \sqrt{1-\gamma}, 1-\gamma)$ and $\boldsymbol{t} = (0,0,\gamma)$ where $0 < \gamma \leqslant 1$.

(h) Now consider $D = (a,a,b)$ and $t = (0,0,c)$. Find the Kraus operators in this case. Show that they reduce to the standard expressions for both depolarizing and amplitude damping channel Kraus operators for appropriate values of a, b, c.

Hint: Mathematica might be helpful but not mandatory.