Problem Sheet 5: Quantum Channels (Part 2) Stinespring and Choi States

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. Write down a Stinespring dilation unitary for:

   a) The completely dephasing single qubit channel (i.e. that channel that kills of all coherence in the computational basis).

   b) The channel the kills of all coherence in the X basis instead of the computational basis?

   c) The channel that prepares an arbitrary mixed state ρ? Describe the general procedure and implement explicitly for a single qubit state with Bloch vector $\boldsymbol{r} = (0.2, 0, 0.2)$.

2. Write $|\text{vec}(A)\rangle$ for the set of Pauli matrices.

3. Prove the following vectorization inequalities:

   i. $|\text{vec}(A \odot B)\rangle = |\text{vec}(A)\rangle \odot |\text{vec}(B)\rangle$ where $\odot$ denotes the Hadamard product (i.e. entrywise multiplication).

   ii. $\text{Tr}[A^\dagger B] = |\text{vec}(A)\rangle^\dagger |\text{vec}(B)\rangle$

4. Use vectorization for prove that the maximally entangled state $\propto \sum_i |ii\rangle$ is invariant under $U \otimes U^*$.

   Although I've not explicitly shown you how to approach the next two questions, everything you need was shown in this week's lecture... try and figure it out yourself!

   (But I will go over how to do this in the lecture next week so if you're really stuck feel free to skip for now).

5. Consider the Choi state $\frac{2}{3}|\phi_-\rangle\langle\phi_-| + \frac{4}{3}|\psi_+\rangle\langle\psi_+|$. Write down a (minimal) set of Kraus operators for this channel. (Are they operators unique?). Hence describe the action (i.e. inputs and outputs) of the channel.

6. Suppose (as in the problem sheet from last week) we have a qubit system A interacting with a qubit environment through the unitary

$$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes X_B \tag{1}$$

   a) Write down the Choi state for the channel induced on the system assuming that the environment qubit starts in the state $|0\rangle$.

   b) Hence write down Kraus operators for the channel (if you found the Choi state from Kraus operators, then deduce another set of Kraus operators from the Choi state).

   c) What are the Kraus operators if instead

$$U = |0\rangle\langle 0|_A \otimes \mathbb{I}_B + |1\rangle\langle 1|_A \otimes e^{-i\lambda t X_B} \ ? \tag{2}$$