Problem Sheet 13: Entanglement Theory (Part 2)

In this problem sheets, and all others, I highly recommend using Mathematica to deal with any messy algebra.

Class problems

1. *Asymptotic entanglement distillation.* Roughly how many singlets are needed to construct 100 copies of the state $|\phi\rangle = \alpha(2|01\rangle - 3|+0\rangle + 5|22\rangle)$, where α is a normalization constant (that you have to determine) and $|+\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$, under LOCC?

2. *Hyperplanes.* In $\mathbb{R}^3$ determine the hyperplanes defined by the vector $\boldsymbol{v}$ for the cases a) $\boldsymbol{v} = (0,1,0)$, b) $\boldsymbol{v} = (0,-2,0)$ and c) $\boldsymbol{v} = (1,1,-1)$. In each case determine the 'positive' and 'negative' sides of the plane.

3. *Superoperators to detect entanglement.*

   Define the maximally entangled state $|\Omega\rangle = |\text{vec}(\mathbb{I})\rangle$ and the Choi state associated with an operator $\mathcal{E}$ as $J(\mathcal{E}) = \mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)$.

   a) Show that $|\Omega\rangle\langle\Omega|^{T_A} = \text{SWAP}$.

   b) Show that $\text{Tr}[A \otimes B\, \text{SWAP}] = \text{Tr}[AB]$.

   c) Assuming that every positive superoperator $\mathcal{E}$ has a decomposition $\mathcal{E}(X) = \sum_i A_i X B_i^\dagger$ deduce that we can define a 'dual map' $\mathcal{E}^*$ such that $\text{Tr}[\mathcal{E}(\rho)B] = \text{Tr}[\rho\mathcal{E}^*(B)]$ for all states ρ and Hermitian operators B.

   d) Prove that

   $$\text{Tr}[(B \otimes \sigma^T)J(\mathcal{E})] = \text{Tr}[\mathcal{E}(\sigma)B] \tag{1}$$

   Let use define the super-operator $\Phi_{\mathcal{J}}(\rho) = \text{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{J}]$

   e) Verify (by constructing a couple of examples) that $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$ and $\mathcal{J}(\Phi_{\mathcal{J}(\mathcal{E})}) = \mathcal{J}(\mathcal{E})$. How does this connect to Eq. (1)?

   f) What is the connection between SWAP and the Peres-Horodecki criterion?

Assessed Problem

1) *Entanglement Witnesses*

a) Define an entanglement witness W, and explain geometrically how it functions.

We want to construct an entanglement witness W that detects entanglement. Consider the hermitian operator $W = \mathbb{I} - \alpha|\Psi\rangle\langle\Psi|$ where $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is a maximally entangled bipartite pure state (proportional to $|vec(\mathbb{1})\rangle$) and $\alpha > 0$ is some number we shall try to fix so that W is an entanglement witness.

b) Explain why imposing the defining property for entanglement witnesses for pure product states will mean it also holds for all separable mixed states.

c) Show that W is an entanglement witness for $\alpha = d = \dim(\mathcal{H})$ i.e. $W = \mathbb{1} - |vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|$. (Hard!)

d) Provide an example state ρ_{AB} for which W detects entanglement

2) *Peres-Horodecki Criterion*

a) State the necessary and sufficient condition for a mixed 2-qubit state to be entangled.

b) Determine whether the state $\rho_{AB} = 0.5|\phi_+\rangle\langle\phi_+| + 0.1|\psi_-\rangle\langle\psi_-| + 0.4|01\rangle\langle 01|$ is entangled or not.

c) Define the Werner state as $\rho = (1-p)\mathbb{I}/4 + p|\psi_-\rangle\langle\psi_-|$. For what values of p is this entangled?

(Mathematica will save you a lot of time here)