Quantum Information Theory

Final exam
Spring term 2024

Assignment date: July 4th, 2024, 9h15
Due date: July 4th, 2024, 12h15

PHYS-550 – Exam – room BCH 2201

- You must answer ALL questions in the short answer section.
- You must answer precisely 2 (out of 3) of the questions in the long answer section.

  Please mark clearly which two you have answered below and **start a new sheet for each of the long answer questions**.
- **Write your solutions in the indicated space.** Scrap paper will not be corrected.
- A simple calculator (without internet access) is allowed.
- Please write your name on the top right corner of each sheet you use.
- Good luck! Enjoy!

NAME STICKER GOES HERE

Short answers: Problem 1	/ 50
Problem A: YES or NO	/ 25
Problem B: YES or NO	/ 25
Problem C: YES or NO	/ 25
Total	/100

Shorter (easier) questions

1. **Partial Trace and Fidelity**

   a) Explain why the partial trace operation is important in quantum theory. (3 marks).

   b) Show that partial trace is a quantum channel by writing it in Kraus form. (3 marks).

   c) Define the fidelity between two states. What is its operational meaning? (3 marks).

   d) Compute the fidelity between the states

$$\begin{aligned} |\psi\rangle_{ABC} &= \frac{1}{\sqrt{3}}(|000\rangle + |110\rangle + |220\rangle) \\ |\phi\rangle_{ABC} &= \frac{1}{\sqrt{6}}(2\,|000\rangle + |010\rangle + |201\rangle) \end{aligned} \tag{1}$$

   (1 mark).

   e) Compute the reduced states ρ_A^ψ and ρ_A^ϕ, of $|\psi\rangle_{ABC}$ and $|\phi\rangle_{ABC}$ respectively on system A. (2 marks)

   f) What is the fidelity between ρ_A^ψ and ρ_A^ϕ? (3 marks)

2. **Measurement.**

   Spock hands you a quantum state and promises that it is either $|\phi_1\rangle = |0\rangle$ or $|\phi_2\rangle = |-\rangle$.

   a) State a POVM-measurement that perfectly distinguishes the states *some* of the time, is inconclusive at others, but never makes a mistake.

   (3 marks).

   b) How would you find the *optimal* POVM elements that perfectly distinguishes the states some of the time, is inconclusive at others, but never makes a mistake? (You don't have to find this POVM but explain how you would find it/prove that it is optimal.)

   (3 marks)

3. **'Measure-and-update' channel.**

   Consider a measure-and-update channel $\mathcal{E}(\rho) = \sum_k |k\rangle\langle k| \otimes B_k \rho B_k^\dagger$ where ρ is the initial state to be measured, $\{|k\rangle\}$ is an orthonormal basis and $\{B_k\}$ are a set of Kraus operators.

   a) Write down the Kraus operators for the channel $\mathcal{E}$. Explain why this is called the 'Measure-and-update' channel. (3 marks)

   Now consider the following choice in operators $\{B_k\}$:

$$B_1 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where λ is real and $0 \leq \lambda \leq 1$.

Further suppose the initial state is the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

b) What is the probability of measuring outcomes 1 and 2? What are the corresponding output states of the system that is measured? (5 marks)

Suppose you now instead have the operators $\{B'_k\}$:

$$B'_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}, \quad B'_2 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

where λ is defined above.

c) What are the probabilities and output states in this case? (4 marks)

d) What do you conclude from comparing your answers in (b) and (c)? (1 mark)

4. **Entropy.**

a) Show that $S(U\rho U^\dagger) = S(\rho)$ for any unitary U. (1 mark).

b) For a composite system AB in a pure state $|\psi\rangle_{AB}$ show that $S(\rho_A) = S(\rho_B)$.

(3 marks).

c) Prove that $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$ (2 marks).

5. **LOCC transformations.**

What deterministic LOCC transformations (one way, two way or no-way) are possible between the states:

a) $|\psi_-\rangle$ and $\alpha\,|00\rangle + \beta\,|11\rangle$.

(2 marks)

b) $|\phi_1\rangle = \sqrt{\frac{1}{14}}(|00\rangle + 3\,|11\rangle + 2\,|22\rangle)$ and $|\phi_2\rangle = \sqrt{\frac{1}{11}}(|00\rangle + |11\rangle + 3\,|22\rangle)$.

(6 marks)

c) With reference to the resource theory of entanglement comment on what you can conclude from your answers to a) and b).

(2 marks)

Longer (harder) questions

Please **pick 2 questions** to attempt - mark your choices clearly on the cover sheet.

Start a new sheet for each question.

Question A - Shot Noise

This question will compare the effect of shot noise for three approaches to estimating $\mathrm{Tr}[\rho H]$, i.e., the energy of a state ρ with Hamiltonian $H = \sum_{i=1}^{M} \alpha_i P_i$ where the P_i are Pauli operators.

1. First suppose you measure each Pauli term individually using N/M shots for each Pauli term. Write down an estimator for this measurement strategy. Show that the estimator is unbiased and compute its variance. Find an upper bound for the variance of the estimator in terms of a norm of the vector $\vec{\alpha}$.

   (5 marks)

2. Next suppose you instead use $N_i = p_i N$ shots for each Pauli term where $p_i = |\alpha_i| / \sum_i |\alpha_i|$. Why intuitively might this be a better strategy than the previous one?

   Write down an estimator for this measurement strategy. Show that the estimator is unbiased and compute its variance. Find an upper bound for the variance of the estimator in terms of a norm of the vector $\vec{\alpha}$.

   (5 marks)

3. Discuss which of these strategies is best for estimating $\mathrm{Tr}[\rho H]$ if only taking into account shot noise? Construct some example Hamiltonians to illustrate your arguments.

   (5 marks)

4. Finally, suppose you instead have a source of classical randomness and with probability $p_i = |\alpha_i| / \sum_i |\alpha_i|$ measure Pauli P_i.

   Again, write down an estimator for this measurement strategy. Show that the estimator is unbiased and compute its variance. And find an upper bound for the estimator in terms of a norm of the vector $\vec{\alpha}$.

   (5 marks)

   Discuss how this method compares to the others? When might you choose to use it?

   (2 marks)

5. Suggest two other methods you might use to estimate $\mathrm{Tr}[\rho H]$. What are the advantages / disadvantages of these methods? (3 marks)

Question B - Matrix Norms

Say you have some arbitrary channel $\mathcal{E}$ and you want to bound $||\mathcal{E}(O)||_2$ where O is an arbitrary observable. You can assume that $\mathcal{E}$ maps a d_S dimensional system to a d_S dimensional system.

We will assume that this channel can be written in Kraus form as $\mathcal{E}(...) = \sum_{k=1}^{M} B_k ... B_k^\dagger$.

You can also assume that the channel has the Stinespring dilation using a d_E dimensional ancilla system.

(Don't be intimidated by the large number of marks for 2 and 3 questions. These are short calculations. But they might require spotting a few tricks.)

1. How is $||\rho_1 - \rho_2||_1$ related to the maximum probability of distinguishing ρ_1 and ρ_2 (assuming you are given ρ_1 or ρ_2 with equal probability)? Prove this relationship.

   (7 marks)

2. Derive the tightest upper bound you can for $||\mathcal{E}(O)||_2$ in terms M, d_S, and the norm of O (you can pick the norm).

   Hint- you may want to derive multiple bounds depending on whether O is a projector or a (sum of) Pauli operators.

   (6 marks)

3. Derive the tightest possible upper bound you can for $||\mathcal{E}(O)||_2$ which is independent of M. It may, or may not, also depend on d_E and/or d_S. It will depend on the norm of O (you can pick the norm).

   Hint- you may want to derive multiple bounds depending on whether O is a projector or a (sum of) Pauli operators.

   (8 marks)

4. Discuss which of these bounds is tighter?

   (4 marks)

Bonus (genuine ongoing research question): what is the tightest bound you can obtain on $||\mathcal{E}^*(P)||_2$ where $\mathcal{E}^*$ is the adjoint of a channel and P is a Pauli? (Note: the adjoint of a channel is not necessarily a channel.)

Question C - Superoperators to detect entanglement

1. Define the maximally entangled state $|\Omega\rangle = |\mathrm{vec}(\mathbb{I})\rangle$. Show that $|\Omega\rangle\langle\Omega|^{T_A} = \mathrm{SWAP}$.

   (2 marks)

2. Show that $\mathrm{Tr}[A \otimes B\, \mathrm{SWAP}] = \mathrm{Tr}[AB]$.

   (2 marks)

3. For any map $\mathcal{E}$ can implicitly define the notion of a 'dual map' $\mathcal{E}^*$ such that

   $$\mathrm{Tr}[\mathcal{E}(\rho)B] = \mathrm{Tr}[\rho\mathcal{E}^*(B)] \tag{4}$$

   for all states ρ and Hermitian operators B.

   For a unitary channel what form does $\mathcal{E}^*$ take?

   (1 marks)

4. Assuming that every positive superoperator $\mathcal{E}$ has a decomposition $\mathcal{E}(X) = \sum_i A_i X B_i^\dagger$. Derive an explicit expression for $\mathcal{E}^*$ in terms of A_i and B_i.

   (5 marks)

5. The Choi state associated with an operator $\mathcal{E}$ is defined as $J(\mathcal{E}) = \mathcal{E} \otimes \mathbb{I}(|\Omega\rangle\langle\Omega|)$.

   Prove that

   $$\mathrm{Tr}[(B \otimes \sigma^T)J(\mathcal{E})] = \mathrm{Tr}[\mathcal{E}(\sigma)B] \tag{5}$$

   (6 marks)

6. We now define the super-operator $\Phi_{\mathcal{J}}(\rho) = \mathrm{Tr}_2[(\mathbb{I} \otimes \rho^T)\mathcal{J}]$

   Prove that $\Phi_{\mathcal{J}(\mathcal{E})}(\rho) = \mathcal{E}(\rho)$ and $\mathcal{J}(\Phi_{\mathcal{J}(\mathcal{E})}) = \mathcal{J}(\mathcal{E})$.

   How does this connect to Eq. (5)?

   (6 marks)

7. Use your answers above to explain the connection between SWAP and the Peres-Horodecki criterion?

   (3 marks)